A STUDY OF GRADE 11 LEARNERS' UNDERSTANDING OF CONCEPTS RELATED TO INFINITY

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DECLARATION

I declare that this Research Report is my own unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

David Neville Merand

_____18th___ day of __August_______ in the year __2014_____ in Johannesburg

ABSTRACT

Do high school learners have a grasp of infinity? Can they use mathematically correct techniques to grapple with issues related to infinite processes and limits?

This study uses the familiar question “Is 0.9 recurring equal to 1.0?” to examine how learners’ prior knowledge and experience influences their answer to this question. Experience relates to discussions and teaching activities in this area during the two lessons examined in this study.

That is, a class of grade 11 learners were given two lessons which highlighted some of the paradoxes associated with infinity and in which techniques which can be used to understand numbers represented as infinitely recurring decimal fractions were discussed.

These lesson were videotaped and transcribed and were then analysed using Sfard’s commognitive framework for thinking and communicating to determine the type of and degree to which learning occurred during the lessons. According to commognition, learning is defined as occurring when a learner changes her discourse on a particular topic in an enduring way.

A set of questions and tasks which probed an understanding of infinitely recurring decimal fractions were used as learning activities and the question “Is 0.9 recurring equal to 1.0?” was used before and after the two lessons to determine if there was any change in learners’ discourse.

In memory of

Dr. Humphrey Uyoyou Atebe

mathematician, lecturer, colleague, friend

4 July 1972 – 5 August 2012

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In addition, I would like to thank the Principal, staff and learners of the school where the study was conducted.

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1 INTRODUCTION

My study investigates the extent to which a group of grade 11 learners have an understanding of concepts related to infinity. The study tries to determine to what extent the learners are able to change their thinking and communicating in order to be able to deal adequately with mathematical concepts related to infinity. The extent to which such changes persist after the interactions is also discussed.

Learners needing to study more sophisticated and rigorous concepts of infinity at later stages of their studies may struggle to succeed if their original exposure to this topic has insufficient rigour. This is often the case since school textbooks and teachers are often constrained in the level of mathematical rigour available for use in the introduction of mathematical concepts such as infinity.

Infinity is one of the core ideas of mathematics. Notions relating to infinity and the manner in which they were resolved were fundamental to the progress of mathematics and remain fundamental to understanding higher mathematics. But, as I elaborate later, the concept of infinity is one that has been struggled with over the course of history – as long ago as approximately 400 B.C. Zeno put forward his paradoxes, and in the late 1800's Cantor grappled with and was repudiated for his ideas related to infinity. And we can see similarly that studies of students' understanding show a similar pattern of grappling. In particular, there seems to be a fundamental shift in thinking required in the move from dealing with finite quantities in mathematics to dealing with the infinite. Because this shift is crucial in allowing access to higher mathematics, I became interested in the process of this change in thinking and under what circumstances, if any, this change occurs in secondary school learners. I thus decided to study a group of high-achieving grade 11 learners as they continued exploring various mathematical concepts related to infinity.

2 IMPLEMENTING THE RESEARCH

I was interested in exploring the way in which learners grapple with the changes when dealing with infinite as opposed to finite numbers; but given the scope of this study it was clearly impossible to deal with all aspects of the infinite and thus I chose to restrict my focus to the topic of infinitely recurring decimals.

The reasons for choosing this topic were:

- It is clearly stated as part of school curriculum
- It is a well-known issue that learners struggle with notions linked to accepting that 0.999…. = 1
- This is particularly interesting as it takes a very familiar discourse in school mathematics (that of decimal numbers) and broadens it to encompass decimals that recur without ending. This broadening of the discourse is particularly interesting because it means a very familiar concept needs to be viewed in a new light and changes to new ideas have to be made. E.g. in the old discourse the "0." in 0.9 signals clearly that the number is less than 1 – however in 0.999…. this is no longer the case.

I conducted this investigation by attempting to have the learners use their existing knowledge and opinions related to infinitely recurring decimals to address various questions about the relationship between infinitely recurring decimals and rational fractions. This is a topic which they have already covered in secondary school mathematics to whatever extent their particular textbook series provides.

The group of learners was chosen from a class of grade 11 students who have chosen to take Advanced Program Mathematics[1]. This indicates a certain aptitude and interest in mathematics which I believed would provide more meaningful data than any other group of grade 11 learners.

The approach I followed was to have an introductory session led by a mathematics teacher who covered various topics, examples and questions related to this subject before moving onto topics related to infinitely recurring decimals. This was intended to re-establish concepts associated with both terminating and non-terminating decimals. The class was then divided into groups and the learners did several exercises related to this topic to reinforce the concepts. Finally they were given some written questions which are somewhat open ended and which required justification for the answers (see Research Design section for more details).

In the second session the teacher uses several different approaches in an attempt to provide a justification and convince the class of the relationship between infinitely recurring decimals and an equivalent terminating decimal fraction representation. These approaches range in mathematical rigour from relatively low to fairly rigorous. The class is given various examples to work through which focus on these concepts. These examples were designed to highlight some of the difficulties encountered when working with infinitely recurring decimals while trying to use terminology and techniques associated with finite decimal expansions. The learners are encouraged to discuss these difficulties in their groups and to attempt to arrive at conclusions related to the "best" definition or meaning of numbers such as 0.9 recurring. Thereafter the teacher led a closing discussion at which the groups are asked to share their findings with the class. At the end of the session the learners are again asked to complete the same written questions as at the beginning of the first session together

[1] This is a matriculation level course offered by some schools in South Africa. It covers more advanced material than the regular mathematics course.

with a justification of their answers. These answers were collected for later review

Both sessions were video recorded and transcribed. These transcriptions form the basis for the analysis related to the research questions described in the subsequent section of the report.

3 RESEARCH QUESTIONS

The area of research in which this study is positioned is that of issues related to learning a new mathematical discourse, namely that of limits and infinity presented in the form of an infinitely recurring decimal fraction. This area is especially significant as the changes required of the learners relate to having a more sophisticated mathematical maturity dealing with concepts that cannot easily be modelled in the finite, concrete environment they are more familiar with. The changes that are required in the learners' thinking are crucial to their subsequent ability to cope with more advanced mathematical concepts such as limits, integrals, convergence of sequences and series and many other advanced mathematical topics.

The broader research problem is that of learning a new and difficult discourse related to mathematical infinity and its related issues. My particular study was related to changes in discourse, how this occurred and how persistent was the change.

This change was studied by analysing whether the learners were able to change their discourse so that they could participate meaningfully in mathematical discourses around infinity given that the only mathematical theory (narratives) and routines that they have been exposed to are those provided by the current textbook (possibly augmented by the teacher's own input).

The specific questions I address are:

1. What changes of discourse occur during or between the lessons in the learners' dealings with infinity?
2. How do these changes occur?

3. When do these changes occur?
4. What opportunities for learning are provided by the teacher and to what extent are these utilised by the learners?

4 LITERATURE REVIEW OF INFINITE IN MATHEMATICS DISCOURSE

In reviewing the literature I have chosen to focus on both the historical development of notions of infinity as well as research studies that chart the development of learners' understanding. I did this because these two approaches both give a good perspective of the struggles that surround the move to the infinite and highlight particular aspects of the discourse that need changing. The changes required were highlighted by the class exercises which I structured to ensure that the learners would be forced to review the arithmetic procedures they were accustomed to using for finite (terminating) decimal fractions and realise that these were no longer adequate for the infinite (recurring) decimal fractions in the exercises.

4.1 Historical Perspective

A review of historical developments in the evolution of the concept of infinity may be useful in charting an individual learner's progression from a naïve intuitive concept to a sophisticated, robust appreciation of the infinite. By tracking the historical progress of mathematical ideas, one may get a better insight into the underlying processes involved in an individual's development. Some reports found that learners' response schemes mirrored those of mathematicians during the course of history when presented with similar problems (Moreno & Waldegg, 1991). Another study, contrasting different approaches found in the use of concepts related to infinity between Korean and English speaking learners, conjectured that these differences were related to the underlying cultural dichotomies (Kim, Ferrini-Mundy, & Sfard, 2012). This latter study did not make the claim that the development of the individual followed the historical path i.e. that ontogeny parallels phylogeny but rather that the development paths of the discourses can differ depending on the context in which they occur. Nonetheless the study of how various concepts developed and matured through the course of time may shed some light on the way learners approach and overcome the hurdles associated with a particular area – specifically infinity and its various aspects in this study.

Aristotle considered the infinite as an on-going process that is never actually attained and in this way was a potential infinity. (Bostock, 1972). This approach has also been identified as the approach followed by school aged children in identifying whether a particular set is infinite or finite regardless of whether the context is numerical or geometrical (Tirosh, 1999). So for example, learners determined by means of an iterative procedure such as adding additional numbers to give the elements of a set or subdividing a line segment, that the given set is infinitely large or contains infinitely many points.

Since the time of Zeno, paradoxes have been generated through seemingly logical uses of mathematical concepts and terminology. The philosopher Zeno of Elea (495-435 B.C.) described a hypothetical situation of a race between a tortoise and Achilles in which he postulates that Achilles can never catch the tortoise who is given a head start, being slower. The reasoning is that by the time Achilles reaches the point where the tortoise started, the tortoise will have moved some distance. By the time Achilles reaches that further point, the tortoise will have moved some more and so on, ad infinitum. Thus Achilles can never catch the tortoise – a conclusion which is plainly wrong based on our understanding of the world (Aczel, 2000). Thus the use of everyday concepts when attempting to deal with either the infinitely small or infinitely large can give rise to inconsistencies and paradoxes. Another common misconception occurs when comparing infinite sets which have the same cardinality (number of elements) where one set is completely contained within another. The misconception is that the contained set must have fewer elements than the containing set. For example, the set of even numbers is completely contained within the set of integers. This argument (discussed further below) which uses everyday concepts related to finite sets often prevents learners from moving from the finite to the infinite case with confidence (Tirosh, 1999).

The development of set theory gave rise to the Cantorian concept of infinite cardinality and ordinality and the ability to compare infinite sets in an explicit way. This led to the concept of 1-1 correspondence (or bijection) being used to compare a set with a proper subset of itself (Moreno & Waldegg, 1991). This generalises the concept of counting for finite sets where each element is paired with an integer (1,2,3,4 etc.) and the number with which the final element in the set is paired is regarded as the total number of elements in the set, that is, its cardinality. In the case of infinite sets what is required is to determine a way of linking each element of one set with exactly one element of another in order to determine that their cardinality (the "number" of elements contained in each) is the same. This is usually achieved by defining a function mapping elements from one set to another and then showing that this function establishes a 1-1 correspondence between the elements. This approach avoids the apparent contradiction where one set is a proper subset of another and yet has the same cardinality as the original set. Once the learner has accepted that this is a valid way of comparing cardinalities, it is possible to begin working with infinitely large sets.

In addition, the way is then open to determine if there are sets with greater cardinality than $\mathbb{N}$, the set of natural numbers, and to compare these higher order infinities. The approach developed by Cantor to do this is discussed subsequently in this report.

4.2 The Role of Intuition in understanding infinitely small and infinitely large

Tirosh (1991) investigated learners' abilities to deal with concepts related to infinity and found that learners' intuitive understanding of actual infinity needed a carefully constructed set of lessons in order to acknowledge and deal with the inconsistencies in their own thinking related to infinity. The study found that a learning unit moving from their intuitive views to consistent mathematically rigorous results needed a "profound knowledge of the students' intuitions towards the specific mathematical theory" (Tirosh, 1991, p. 214). Thus for example in order to progress from finite to infinite sets lessons were constructed which used finite sets as the starting point for extrapolating to infinite sets. Students were given problems related to comparing the number of elements within finite sets using several principles, namely counting, 1-1 correspondence, part larger than whole etc. They were then asked to consider which of these methods could be transported to the case of infinite sets and to consider any issues or problems that may arise from using them in the new context. The idea was to reveal some of the inadequacies of the techniques used for the finite case and to attempt to justify Cantor's choice of 1-1 correspondence when comparing the cardinality of sets (Tirosh, 1991, p. 207). This study concluded that teachers need to be cognisant of the pitfalls associated with using intuitive methods and need to show ways in which these traps can be avoided by exploration rather than by merely presenting formal definitions.

Attempts to gain an appreciation of the infinite by extending one's conceptualisation of the finite can be fraught with difficulties. Tall (2001) refers to such extensions as "natural infinities" and goes on to discuss potential problems and contradictions associated with such an approach. A natural approach is when an individual uses conceptual imagery to give personal meaning to a formal definition. This can then be used to engage in thought experiments to arrive at a better understanding of the area being studied. A formal approach is used when formal definitions are used to develop theorems and strategies to prove these theorems. Tall explains that most mathematicians use both approaches whereas many students prefer to rely on the natural approach only, using a "personal version of the definition, sometimes inadequate, sometimes distorted, with the result that there was a broad spectrum of success and failure" (Tall, 2001). This explains how a learners' intuitive or prior knowledge can result in difficulties when trying to extend the mathematical discourse across the boundaries of finite to infinite. Tall found that learners often rejected the concept of infinite cardinal numbers. However using prior experience to reconstruct knowledge in the face of contradictions can be beneficial. The principal that "the whole is greater than the part" is true for finite sets but not for infinite sets. In fact, this difference can be used as a definition of an infinite set. This is described in (Tall, 2001, p. 204) as follows: "a set is defined to be infinite if and only if it can be put into 1-1 correspondence with a proper subset of itself ". This both acknowledges and resolves the contradiction.

4.3 Other Influences on understanding Infinity

In Tall, Smith, and Piez (2008, p. 217) an attempt is made to explain how an exposure to the technological world and the representation of numbers in computers can affect the understanding of infinity. Computers inherently have a finite representation of numbers which is independent of the base used for representation or the size of the storage unit used for a number.

A computer thus has a "smallest" positive number which it can represent. This number represents the difference between 1 and the number closest to 1 that the computer can represent. This mirrors the (incorrect) argument used by some of the students in this study that there is a small (infinitesimal) difference between 0.999… and 1 because no matter how far you go to the right of the decimal place, there is still a small difference between the two numbers. This "proves" that these are two distinct numbers hence $0,\bar{9}$ cannot equal 1. This argument is examined during the lessons in this study as a potential reason for a learner rejecting the conjecture that $0,\bar{9}$ is or must be defined as 1. This viewpoint remains firmly entrenched in some of the learners' minds even after the lessons and is supported by an extensive number of websites propounding the same viewpoint i.e. that 0.9 recurring is not = 1. (http://www.debate.org/debates/0.99-Repeating-does-Not-equal-1.00/1/)

4.4 Potential and Actual Infinity

Various studies have explored two different aspects of infinity, often referred to as *potential* infinity and *actual* infinity. This is described by Arzarello, Bartolini Bussi, and Robutti (2004) as being either an infinite process which never terminates (potential infinity) or as a particular object (actual). They describe how historically various approaches have resulted in conflict between these alternate views of infinity. Thus, for example, the use of the "infinitesimal" in the development of calculus by Leibniz and Newton was characterised by having this construct considered sometimes as actual object, albeit infinitely small and then at other times as "fictions useful to abbreviate and speak universally" (Edwards, 1979, p. 264) in (Kleiner, 2001).

The concept of a Basic Metaphor of Infinity (BMI) is proposed by Lakoff and Nunez (2000, p. 159) whereby the concept of actual infinity comes about as a result of the iterative process of potential infinity. The approach adopted is to consider a source and target domain and an iterative process which has at each step an initial and a resultant state. The effect of the metaphor is to add the completion of the process to the target domain and thus move from a potential to an actual and final result of the process. Thus the notion of infinity changes from “an open-ended process into a specific, unique entity” (Arzarello et al., 2004, p. 90). An example of this is the situation when determining the area underneath a curve by means of a finite partial sum (Riemann sum). By considering greater numbers of rectangles we come to the result that the area under the curve (an infinite sum) is the limit of an infinite sequence of partial sums.

Tall (2001) discusses how an individual may be exposed to other forms of infinity which he calls “formal concepts of infinity, built from formal definitions and deductions”. The concept of a “potential infinity” representing an ever closer approach to a result is discussed using limits as an example and culminating in the definition of a “procept” whereby a process and the outcome of that process are represented by the same symbol (ibid., p. 233). This study uses similar concepts in the discussions of how, say, the limiting process of $0,\bar{9}$ leads to a definition of this as being 1, primarily so as to achieve consistency of results. It is precisely this never-ending process of convergence which can prevent the individual from making the leap to an object which allows a realisation that the only possible object representing the limit in the case of $0,\bar{9}$ is, in fact, 1.

This approach to regarding the result of a process as an object in its own right is also embodied in the principal of reification (Sfard, 2008, p. 101) which is covered in more detail in the Theoretical Framework section. In this situation the use of a noun denoting the object replaces the uses of a verb denoting the process. Thus for example, the limiting process of approaching a particular value to an ever-closer degree is replaced by *the* limit. This scope of this study did not encompass the concept of limit *per se* so checking for reification as defined in the theoretical framework was not explicitly performed.

4.5 Issues with the infinitely large

Early philosophers and mathematicians had largely rejected the concept of an actual infinity and remained committed to a potential infinity which is never reached. So Aristotle stated that

> "the infinite is potential, never actual"

Gauss in 1831 complained about

> "the use of an infinite quantity as a completed one"

and even as late as the twentieth century Poincaré stated

> "there is no actual infinity, and when we speak of an infinite collection we understand a collection to which we can add new elements unceasingly.

All these quotes are cited in Tirosh (1991).

We thus see continued use of infinite processes rather than an attained actual being discussed. The difficulty associated with the actual infinity was well known as far back as 1638 when Galileo observed that if an actual infinity was admitted then there would be as many natural numbers as perfect square numbers. This conflicted with the concept of a proper subset containing fewer elements than its containing set. (Tirosh, 1991, p. 200). Bolzano in the early 1800's extended this to the continuum of real numbers by concluding similarly that the real numbers between 0 and 1 could be put into a 1-1 correspondence with the real numbers between 0 and 2 and that there are thus the "same number" of elements in these two sets (Aczel, 2000). It remained for Cantor to show subsequently that there is a difference between these two categories of sets (denumerable or countable in the case of natural numbers and uncountable in the case of the real numbers).

4.6 Issues with the infinitely small

Calculus as originally developed in the 1670’s by Newton and independently by Leibniz used the intuitively appealing concept of infinitesimally small items, in fact arbitrarily small. This was successfully used to develop an extremely powerful set of tools for studying the physical world. But yet again these concepts dealt with a potential and an infinite process i.e. an amount becoming arbitrarily small. The actual rather than a potential required the concept of an attained limit as the actual result. The use of infinitesimals was rejected subsequently due to philosophical objections. The rigorous treatment of calculus was achieved in the 1870’s using an approach perfected by Weierstrass based on the standard ϵ , δ definition of limits (Keisler, 1986).

A great amount of controversy existed around the lack of rigour in the use of infinitesimals during the development of calculus. Indeed philosophical issues were raised by Bishop Berkeley in 1734 in which he decries the use of these “neither finite Quantities nor Quantities infinitely small, nor yet nothing” (Edwards, 1979). The concern was that by using these vague concepts, a new science of truth was being developed. In fact, in 1974 the Berlin Academy offered a prize for an explanation of how so “many true theorems can be deduced from a contradictory supposition (being the existence of infinitesimals)” (Kleiner, 2001).

The use of this technique of considering arbitrarily small items as being “actual” i.e. used in algebraic calculations as non-zero items and then being disregarded or discarded as being arbitrarily small (i.e. potentially zero) continued to be used by engineers and physicists because they give rise to correct results. Pure mathematicians however, regarded these techniques as somewhat suspect until the development of non-standard analysis by Robinson resulted in a mathematically rigorous approach to the use of infinitely small and infinitely large numbers (Robinson, 1966).

Infinitesimals had been excluded from mathematics as being unsatisfactory from a logical standpoint but were now placed on a rigorous footing and were re-established as a result of this work. This provides an alternative approach to the teaching of concepts related to infinity such as limits where the intuitive concepts of something being “arbitrarily close to” i.e. “small distance away from” can be used as opposed to the algebraically rigorous concepts demanded before.

Thus where Leibniz identified “2x + dx” as 2x, Robinson would write st(2x+dx) = 2x where st() represents the standard part of a hyperreal number (Kleiner, 2001, p. 169). A hyperreal number consists of a standard part and an infinitesimal part. The infinitesimal part is “closer” to zero than any other real number. Thus the implicit has become explicit and the need for limits is removed. My study uses similar intuitive approaches for infinitesimals to address the issues of a recurring decimal becoming arbitrarily close to a particular non-recurring decimal and thereby being essentially the same (in the potential infinity sense) or literally the same (in the actual infinity sense).

As an example of the lack of rigour in existing mathematics textbooks, Goba and Van Der Lith (2008, p. 4) present an intuitive (and widely used) approach for converting from a recurring decimal to a rational fraction. This technique involves multiplying the decimal by a suitable power of 10 and solving the resulting set of simultaneous equations. This algorithm or more precisely, mathematical routine, in the terms of Sfard (2008) ignores the issue related to multiplying a recurring decimal where “carrying” is required such as when $0.\dot{3}$ is multiplied by 4. The difficulty occurs when performing the multiplication

0.33333333333333333333333333………..

x 4

The difficulty occurs with where to begin multiplying by 4. The conventional algorithm multiplies the rightmost digit with 4 and then continues to the left. But in this case there is no rightmost 3 as the list of 3’s continues infinitely to the right. This difficulty is often glossed over by choosing examples in which this problem does not occur such as when $0,\bar{3}$ is multiplied by 2.

My study explores the anomaly such an approach causes and uses it as part of an attempt to disrupt intuitively held concepts related to finite arithmetic which do not transfer to the infinite case. This approach is used as part of the lessons and exercises presented to the learners.

4.7 Cantor and the development of Actual Infinity

Cantor was the first to use the concept of 1-1 correspondence to distinguish between the sizes of infinite sets. In 1873 he showed that the set of algebraic numbers (roots of polynomials with rational coefficients) is countable i.e. can be put in 1-1 correspondence with $\mathbb{N}$ the set of natural numbers (Jahnke, 2001). He also showed a similar relationship between the rational numbers $\mathbb{Q}$ and the natural numbers $\mathbb{N}$.

Cantor found in 1874 that the same cannot be said of the real numbers. This he achieved by assuming that a 1-1 correspondence exists between the natural numbers and the real numbers and deriving a contradiction from this assumption. The following approach which he used is adapted from Wallace (2010).

Assume that the real numbers are each written in decimal notation and that every terminating decimal is written as a non-recurring decimal – thus 8.5 would be written as 8.49999999… and 0.137 as 0.1369999999…. . This means that each real number will appear only once. Arrange the real numbers in a list where the position in the list provides the 1-1 correspondence with the integers (i.e. the third number in the list is matched with the integer 3). The axiom of choice guarantees that an ordered list of real numbers can be created and that this ordered set will have a first element. Now we consider for each number in the list the digits to the right of the decimal point (i.e. the fractional part of each real number). The digits to the left of the decimal point can be ignored. We then construct another real number which is not in the list as follows:

> Suppose X is the real number we are constructing. As its first digit after the decimal point consider the first digit after the decimal point of the first number in the list. Suppose that digit is “a”. Take as the first digit of X “a-1” if “a” is 1 through 9 and take it as 9 if “a” is 0. The important point to notice is that X now differs from the first number in the list in at least one position i.e. the first position.
>
> For the second digit of X after the decimal point consider the second digit of the second number in the list. Apply the same process as before to derive the second digit after the decimal point of X. Note that X now differs from the second number in the list in at least one position i.e. the second position.
>
> Continuing in the same fashion with every number in the list creates a real number X which differs in at least one position from every number in the list and hence is not in the list.

Thus we have created a real number X which is not in the list which is a contradiction showing the set of real numbers cannot be put into a 1-1 correspondence with the set of integers.

Cantor thus proved by contradiction that the set of real numbers is not countable and hence there must exist transcendental numbers (Aczel, 2000; Jahnke, 2001). In fact he discovered a whole hierarchy of transcendental numbers by showing that the power set of a set (set of all subsets) has cardinality greater than the set itself. By this means he created actual infinities which could be manipulated, studied and used as objects in their own right. In fact Cantor developed an arithmetic of cardinal numbers.

4.8 The role of teacher knowledge and experience

Teacher knowledge and experience has an important role to play in teaching an appreciation of infinity which is robust and can withstand the strange contradictions which can occur if a naïve extension of finite mathematics is used.

I reviewed the literature related to studies similar to mine in order to determine the approaches taken by others and to review the types of arguments and examples used by others in helping learners understand the issues associated with moving from the finite to the infinite, and more particular, in dealing with infinitely recurring decimals in a way consistent with their existing finite mathematical concepts. I also used the structure of some of the worksheets and exercises in these studies to design my own pre- and post-lessons questions for the learners as well as the exercises they were asked to perform. This ensured that the exercises served both as a revision of the concepts as well as an opportunity to highlight difficulties with arithmetic procedures when moving from finite (terminating) to infinite (recurring) decimals.

Tsamir (1999) examines a situation where prospective teachers are presented with problems related to comparing the number of elements of firstly finite and then infinite sets. She found that in the group of prospective teachers who had some introductory training of Cantor Set Theory plus an exploration of intuitive ideas, there was a greater appreciation of the need for consistency in mathematical results and the consistent use of a single method of comparison (1:1 correspondence) as opposed to various solution methods used by the other groups of teachers which lead to contradictory results. I used this need for overall consistency in mathematics when designing the arguments used in the lesson plan for the teacher used in my study.

Problems associated with representations used for the teaching and learning of infinite series (of which infinite recurring decimals are an example) were studied in Seffah and Gonzalez-Martin (2011). They found that virtually all textbooks surveyed did not use both algebraic and graphic representations of series, did not coordinate the algebraic and graphic registers and did not have tasks requiring students to provide a visual interpretation of the concept of a series. They conclude that this could have implications for the difficulty students find in learning and applying these concepts. These ideas were used in the lessons where both visual and algebraic representations of infinite series were used in the explanations.

A study described in Brijlall, Maharaj, Bansilal, Mkhwanazi, and Dubinsky (2011) showed that pre-service teachers had reservations about stating that 0.9 recurring is equal to 1 and that by taking them through various exercises and discussions this view was revised. The program involved a structured set of activities related to representation of rational fractions as infinitely occurring decimals, operations on these items and comparison of results. The approach was to consider 0.9 recurring as a sum of 2 other infinite recurring decimals and to explore the implications of this. So for example consider 0.6 recurring and 0.3 recurring. What are their corresponding fractional representations? What occurs if you add them together? At the end of the program, more learners were convinced and could justify their decision that 0.9 recurring is the same as 1. The exercises in my study used similar examples to develop the concepts needed to perform arithmetic with recurring decimals.

Conradie, Frith, and Bowie (2009) describe a scenario in which pre-service teachers were presented with 4 different approaches in an attempt to convince them that 0.9 recurring is equal to 1. These vary in degrees of mathematical rigour from low to high and also attempt to show that mathematical consistency is best achieved if they are *defined* to be equal. This approach was used as the basis for the lessons which formed part of this study.

The approaches used in the study and in the lesson were as follows:

Approach 1:
Algebraic: Define an unknown (say x) to be the recurring decimal, create a set of simultaneous equations and then perform algebraic manipulations to solve for x.

Approach 2:
Establish an equality relationship between an accepted recurring decimal and a rational fraction (e.g. $\frac{2}{3} = 0.\bar{6}$) and perform manipulations yielding the desired result.

Approach 3:
Represent the recurring decimal as an infinite geometric series. Solve using the formula S = $\frac{a}{1-r}$

Approach 4:
Consider all proper fractions with denominator 9 and hence conclude the desired result.

$\frac{1}{9}$ = 0.1111…

$\frac{2}{9}$ = 0.222….

And so on giving finally

1 = $\frac{9}{9}$ = 0.999…..

5 THEORETICAL FRAMEWORK

The theoretical framework chosen for this study had to be well-suited to an analysis of the research questions in the study and provide the tools needed to focus on the core issue under scrutiny which was the struggle experienced by learners in moving from a mathematical discussion of finite or terminating decimal fractions to that of infinite or recurring decimal fractions. In order to do this I needed tools which could assist in several dimensions associated with the problem such as representations, procedures, discourses, and teacher-learner interactions. Below follows a description of the framework chosen and some of the various components of this framework together with examples related to the particular mathematical topic being studied, namely, infinitely recurring decimal fractions.

This study uses Sfard's commognitive framework as the theoretical lens for analysing the data that emanates from this investigation. The primary reasons for choosing this particular theoretical framework are:

1. It provides practical analytic constructs for determining learners' ability to participate effectively in the mathematical discourses at hand.
2. It utilises various well-defined constructs for analysing the mathematical nature of the interactions between participants and the teacher.
3. It highlights the teacher-learner agreement being forged as it occurs and shows the leading discourse being established and developed by the teacher.

These points will be further clarified and discussed later in this chapter.

The commognitive framework arose out of a need to be able to understand and analyse the processes occurring within the classroom and, in particular, the mathematical classroom. Sfard (2007) describes how this became feasible and desirable as a result of the improved ability to record the minutia of activities as and when they occurred aided by the proliferation of video and audio recording technology. As a result, research began to focus not only on broadly based results (the *what*) related to various teaching and learning approaches but also on the way in which changes in teaching and learning were taking place (i.e. the *how*).

According to Sfard (2007), the traditional acquisitionist approaches (encompassing both active and passive acquisition of knowledge) have been replaced by more participationist understandings which attempt to explain the interpersonal dynamics occurring between learners and the more expert practitioner (the teacher in this case). Thus concepts and terms such as “communities of practice” (Wenger, 1998) have become used to describe how an individual participates within a particular environment and becomes more proficient over time with the assistance of a teacher or expert. In this way, personal expertise and learning becomes an individualised version of a collective experience or activity.

The commognitive framework seeks to determine what supporting environment is needed or required for this internal change to occur and also how new and fundamentally different concepts can be integrated with existing knowledge. My focus in this study is on mathematical concepts but the framework is also applicable in other areas.

The commognitive approach developed by Sfard unites the way this internal change occurs by combining both the cognitive aspects of knowledge acquisition together with the communicational aspects related to inter- and intra-personal interactions. This approach is combined into the term commognition to convey the idea that thinking (individual cognition) and communicating are different manifestations of the same phenomenon (Sfard, 2002, p. 296).

The commognitive approach characterises thinking as a particular form of communication which is an individual activity rather than a group one (intrapersonal rather than interpersonal) and thus “thinking is an individualised version of (interpersonal) communicating.” (Sfard, 2008, p. 81).

5.1 Mathematical Discourses

The communication which takes place within a setting such as a mathematics classroom optimally takes place according to a set of accepted rules defined and agreed to by all participants according to the subject area under consideration. These rules form the *discourses* for a particular subject area and divide society into “communities of discourse” (Sfard, 2007, p. 573). Individuals become a member of this community of discourse if they are able to participate in communication within any community which practises this discourse – which is not limited to discourses which are defined as mathematical.

In this context, Mathematics is regarded as a particular form of communication which is accomplished by means of discourses, each with its own set of objects and processes operating on these objects. There is, however, a significant difference between mathematical discourses and those from other branches of science. This difference is best understood by considering that the objects associated with other branches of science (e.g. physics or chemistry) are generally concrete and pre-existing whereas those in a mathematical discourse are inherently highly abstract and usually have layers of abstraction. The objects of a mathematical discourse have to be created within the system and thus mathematics is referred to as an autopoietic system and so is “ a system that contains the objects of talk along with the talk itself and that grows incessantly ‘from inside’ when new objects are added one after another” (Sfard, 2008, p. 129).

A particular vocabulary is developed with very specific meanings for words which have a pre-existing everyday meaning (e.g. round, sphere etc.) but often with a much more specific and precise meaning than in everyday use (e.g. limit). “Doing mathematics” thus becomes participation in a mathematical discourse according to the accepted rules of the discourse. These rules may vary depending on the community in which the discourse takes place; thus a discourse in a junior school mathematics classroom may not continue to be acceptable in a senior school mathematics classroom.

An example of a discourse which may change at different stages of a learners’ school career is that related to the concept of *multiplication*. The discourse may originally be constructed around the concept of repeated addition of a number, i.e. adding multiple copies of a number together to give a result such as “multiply 3 by 4” meaning take 4 copies of 3 and add them together giving 12. As learners progress to working with rational numbers and operations with fractions, the view can be changed to include proper fractions as the first term. E.g. “multiply 3.5 by 4” can be regarded as “take 4 copies of 3.5 and add them together” giving 14. The discourse has to change though to include examples such as “multiply $\frac{1}{2}$ by $\frac{1}{4}$ “ as a way of taking $\frac{1}{4}$ copies of $\frac{1}{2}$ is now required. The view that “multiplication makes the result bigger than the component parts” which is an outcome of working with integers must also change when working with rational fractions where one or more of the terms is less than 1. For my particular study, in which the infinite is involved, this discourse has to be reviewed even more significantly because when trying to multiply infinitely recurring decimals where "*carrying*" is required, it is not clear whether one can even talk about the topic in the same way.

Thus learners need to modify their use of everyday words to align with the specific use within the particular discourse in which they are attempting to participate. In fact, differing use of the same words when attempting to extend a discourse for example, as described above, can give rise to *incommensurable discourses* – and this apparent clash and difficulty in communicating between and within participants gives rise to a major opportunity for the possibility of learning. This situation is discussed further in the Analysis and Findings section of my study.

The vocabulary used in a particular discourse is often based on an everyday concept which has been extended and for mathematical discourses in particular, made more rigorous. An example which is particularly pertinent for my study is that of the concept of *limit*. When used in the everyday sense it may have a connotation of restriction, boundary or control. The mathematical discourse around limit is a much more complex structure which can be defined to various degrees of rigour depending on the particular requirements and sophistication of the community of discourse. This may range from the classical epsilon/delta definition used for defining continuity of functions to the less rigorous definitions of limit used for sequences and series using the concept of getting *arbitrarily closer* to a particular value. It is this latter concept and vocabulary that we will be using in this study when attempting to establish that the recurring decimal $0.\bar{9}$ is arbitrarily close to 1 when considered as the limit of an infinite series and so is actually the same number.

In order to analyse the discourses making up my study and to be able to decide whether a particular discourse is mathematical, I focused on what Sfard (2008) has described as the major constructs or characteristics used in a discourse to effectively manage communication. The major discourse characteristics which are considered are: *visual mediators, narratives, routines* and *words* (Sfard, 2008, p. 133). These will be defined and discussed further in the subsequent sections.

5.2 Visual Mediators

Visual mediators, as their name implies, are constructs which are used by "participants of discourses to identify the object of their talk and coordinate their communication " (Sfard, 2007, p. 573). In this context a discourse may be either inter or intra-personal. These mediators are images which enable processes related to the mathematical objects to be conceptualised and examined. Colloquial mathematical discourse may make use of concrete visual mediators such as the actual physical objects used when counting a set of items or by using actual bank notes when performing a financial transaction. Concrete visual mediators may still be imaginary when used for example for visualising in an intrapersonal activity. Concrete visual mediators exist independently of the discourse in which they are a part.

In contrast, a more general class of visual mediators known as *symbolic artefacts* which are imaginary i.e. created solely for use in the mathematical discourse can be used to represent the mathematical objects of the discourse. These symbolic artefacts include *algebraic symbols* as well as *icons*. Note that these artefacts are regarded as part of the thinking process rather than as external enablers.

Icons are symbolic artefacts such as diagrams, sketches, graphs and drawings around which a discussion of various (often spatial) aspects of a mathematical object can occur. Some examples are: using a graph of a function to illustrate aspects of slope and gradient when discussing derivatives, or to illustrate zeroes of a function or to find the solution of simultaneous equations by means of the intersection of the lines representing each equation. All of these are diagrammatic representations of the underlying mathematical objects which are used to accentuate particular properties of or relationships between the objects by means of visualisation.

Algebraic symbols are another form of symbolic artefact used in a discourse when algebraic manipulation is more effective or convenient – for example when factorising an expression to find its roots. As with all symbolic artefacts, algebraic symbols are non-concrete visual mediators used to facilitate discussion or reflection on a mathematical object. Part of the goal of schooling is to have learners gain fluency in performing activities involving the more general algebraic symbols and operations instead of solely using numbers (Sfard, 2008).

In my study an example of a critical visual mediator is the algebraic symbolic representation of the recurring decimal $0.\bar{9}$. Using this representation of a zero to the left of the decimal point followed by an infinite string of 9's to the right of the decimal point visually supports the argument that $0.\bar{9}$ must be less than 1 because learners are accustomed to recognising that a decimal with a leading zero followed by a decimal point (or comma) is a decimal fraction less than 1. Thus a major obstacle to the learning which can occur in the lessons is exactly this visual mediator which serves to hinder the objective of the lesson. Part of my analysis was to examine to what extent the learner with the help of the teacher is able to surmount this obstacle.

Visual mediators are often used to assist in the understanding of a particular discursive object. Thus a particular signifier may have several realisations which can be used to emphasize or explain different aspects of the mathematical object. The different symbolic artefacts used to identify the signifier may be interchangeably used and thus a realisation tree of the particular object could have one of the realisations as its root.

Thus a signifier such as "The solution of $7x+4=5x+8$ may have a realisation tree consisting of a table of values, a graphical image or algebraic manipulations (Sfard, 2008, p. 165). These signifiers and realizations are often used interchangeably. A signifier can be thought of as "the question" and the realization as "the result".

5.3 Narratives

Narratives refer to written or spoken words or text describing an object or a relation between objects. Narratives may be accepted as true (*endorsed*) or rejected by participants in the discourse using discourse-specific substantiation procedures. Note that what may be regarded as an acceptable substantiation process in one situation may not suffice in another. So, for example, at school level a less rigorous procedure may be acceptable whereas in a university situation it may not. Narratives which are consensually adopted are often referred to as theories and where mathematical, will contain constructs such as definitions, theorems and proofs. Narratives may be defined at the object level which describes mathematical objects such as equations and variables; or at the meta-level which describes the discourses themselves e.g. agreed approaches about what constitutes a proof and how a proof may be constructed. (ibid. p.574). Narratives describe not only the objects of a discourse but also the procedures involved in manipulating the objects of a discourse.

An important endorsed narrative in my study is the theorem which results in a simple formula for the sum to infinity of a geometric series. This is used to find a representation for $0,\bar{9}$

$S_n = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10^{n-1}}$ This is a geometric series with first term

$a = \frac{9}{10} \textit{ and ratio } r = \frac{1}{10}$ giving a sum to infinity of $\frac{a}{1-r} = 1$

This is an endorsed narrative because the class has already been exposed to this formula and has used it in previous work, albeit in a totally different context than recurring decimals.

In the lesson the teacher establishes the relationship $S_1 < S_2 < S_3 < ... < 1$ or equivalently, 0,9<0,99<0,999<…..<1. She also establishes that $0,\bar{9}$ represents the sum to infinity of this series which is thus 1 by the above formula.

5.4 Routines

Routines refer to repetitive patterns adopted by participants in a discourse when manipulating or processing mathematical objects. They may include algorithms for performing particular processes e.g. long division, or they may be ways of categorising mathematical objects, comparing different situations, moving from one mathematical realm to another (e.g. when transferring a problem from one mathematical domain to another which has more tools suited for attacking the problem at hand). Routines are governed by rules which may be *object-level* rules controlling accepted ways of operating or they may be *meta-level* rules which govern the discourse itself – such as what constitutes a valid proof in a particular domain. Routines which are acceptable in one situation (e.g. school level) may be unacceptable in another (e.g. college level). Routines have two distinct aspects:

- The *how* of the routine which describes the procedural actions to be used when invoking a routine
- The *when* of a routine which governs the applicability of the routine i.e. the situations under which it is appropriate to use the routine (Sfard, 2008, p. 220)

In determining the proficiency of learners I consider both the *how* component of a routine being successfully utilised; and in order to verify a deeper understanding of the discourse I also needed to consider the *when* component of a routine. In other words, is it being used appropriately in the given situation?

The well-known routine for adding two numbers in terminating decimal representation is an example of a routine used during the lessons which form part of this study. This is an object-level routine as it involves manipulations on the underlying mathematical objects, namely decimal numbers. An aspect of the use of this routine which I look at in my analysis is whether the learners ever realise that in the initial examples in the worksheets they extend the use of this routine from one domain to another i.e. from terminating to recurring decimals without verifying its validity. Choosing different examples for which this routine breaks down is what gives rise at the end of lesson 1 to a commognitive conflict (see definition below).

5.5 Words

Word usage governs the vocabulary used to communicate the mathematical ideas in a discourse. These key words usually signify quantities and shapes and generally have a more precise meaning than in general everyday use. Word use is critical because it "is responsible for what the user is able to say about (and thus to see) in the world" (Sfard, 2008, p. 133). Word usage can indicate the degree to which objectification has occurred for a learner. Thus, for example, if number-words such as one, two, three are used exclusively as adjectives in connection with some noun and not as a self-standing noun, this indicates that objectified use of number has not yet occurred. Sfard & Lavie (2005) show how learners' early number use is often a ritualized form of counting-chanting prior to the objectifying of the number concept. The learners are thus unable to identify the "sameness" of groups (sets) of different objects which have the same cardinality. In fact, the point is made that once this transition has occurred, it is difficult for the individual to reverse the situation and hence it may be impossible for the teacher to appreciate learner difficulties in this type of situation.

The area of recurring decimals has its own specific vocabulary, thus even the word recurring needs to be clearly understood by all participating in the discourse. It must be agreed that the meaning of infinitely recurring means literally never ending, not to be confused with going on and on for a very long time. In my analysis I look at the implications of this for the learners in being able to move from the concept of an infinite process to the result of this process, i.e. the attained limit.

5.6 Object- and Meta-level learning

In a commognitive framework learning is characterised as a change in mathematical discourse and this change occurs when a learner changes or is forced to change a particular mathematical discourse, firstly to cater for additional scenarios (e.g. when extending fractional arithmetic to deal with numerators which are larger than the denominator) and secondly when the rules of the discourse break down and fail to work in the situation under consideration. An example of this is when trying to add two recurring decimal fractions which involve "carrying", for instance $0,\bar{4}+0,\bar{7}$.

The first of these is known as *object-level* learning and is an extension of an existing discourse to cater for additional situations, an extended vocabulary or the addition of new endorsed narratives such as proofs of additional theorems.

The second of these is known as *meta-level* learning and occurs when there is a breakdown in the existing discourse and new rules or meanings need to be negotiated by the participants (or the individual if this is an intra-personal discourse). For example, this type of change occurs when learning that multiplying two negative numbers gives a positive number.

This breakdown is known as a *commognitive conflict* and is often manifested by participants endorsing conflicting narratives. If the discourses differ in their use of words, narratives or routines they are termed *incommensurable* as, instead of one being correct and the other incorrect, they rather cannot be compared. An example of this is the situation that occurrs in my study where the learners perform addition of recurring decimals by extending the routine for adding decimal fractions which results in different groups getting different results because of issues involving the "carrying" of values. Another example which occurrs in this study is the use by the teacher of various mathematical arguments showing that $0.\bar{9}$ =1 by using existing endorsed narratives to bring about a commognitive conflict with learners' belief that it is less than 1. This belief is supported by the visual algebraic representation used for the recurring decimal.

The opportunity for meta-level learning results when the learner is able to accept and individualise the changed discourse of the (generally) more expert interlocutor (Sfard, 2007, p. 578). This usually occurs when a teacher alters the discourse in a way that would not typically have occurred to the less experienced learner.

5.7 Reification

Reification of mathematical objects occurs when verb-based utterances are replaced with nouns. Thus instead of referring to what must be done, the learner refers to the result of the process as an object itself (or noun). For example, "adding 3 to 2" will be replaced by "the sum of 3 and 2". This allows the learner to disregard the processing and to consider the result of the process independently of the process itself. As another example, the signifier 5/7 can be considered as "take 5 and divide it into 7 parts" (process) or instead , as "5/7 of the whole"(object) (Sfard, 2008, p. 171). This particular transition becomes crucial when moving from *potential* infinity involving a process of getting ever closer but never attaining a particular result to a form of *attained* infinity as the final result which can then be used independently of the process used.

In order for learners to accept that the infinitely recurring decimal and the rational number which it approaches are equal, for example that $0,\bar{3} = \frac{1}{3}$, they will have to be able to perform the activity of *saming* which involves equating different realisations of the same signifier. Thus they will need to accept that an infinitely recurring decimal is merely another realisation of the signifier rational number. Sfard (2008, p.154) defines this as “A realization of the signifier S is a perceptually accessible thing S# so that every endorsed narrative about S can be translated according to well defined rules into an endorsed narrative about S#.”

5.8 Use of Commognitive Framework

The commognitive framework described in this section has been used in areas of mathematical education research similar to my own and published in peer-reviewed journals. I will now discuss some of these studies.

A study comparing the impact of language on students’ understanding of infinity was conducted using English- and Korean-language learners (Kim et al., 2012). In this study the authors attempted to gauge the extent of learners’ experience of language on their ability to develop discourses on mathematical topics, in this case, an appreciation of infinity. They explored the concepts of “boundlessness”, “limitless”, “endless” and similar concepts as expressed in Korean (often with Chinese origins) and also in English. They found that the Korean students had a more formal exposure to words relating to infinity as their language did not have the colloquially similar uses found in English which used words like “infinitely”, “without limit” etc. in everyday speech. They found that this prior exposure resulted in a more process-oriented view amongst the English-speaking students when compared to the Korean-speaking ones. This resulted in the latter being more adept at speaking in a way superficially closer to the formal mathematical discourses on infinity.

Another study used this framework to compare the differences between the mathematical equivalence of a written curriculum and its enactment in the classroom (Newton, 2012). In conducting this project the author considered these two components as discourses and compared their realisations i.e. icons, symbols and concrete objects in terms of the relative prominences given to each. She wished to determine whether the mathematical object in the curricula (in this case rational numbers as symbolised by fractions) where sufficiently similar to one another. She found that the word use differed in the two areas; for example, the written curriculum used the word *fraction* in a more abstract way than the enacted curriculum. In addition, the endorsed narratives were more algorithmic in nature than the enacted curriculum. In addition the realisations of fractions were often different, for example number-line representations were absent in the enacted and present in the written curriculum. The author concluded that these differences may impact a student's mathematical competence.

A study which is closely allied to my present report is presented in (Gucler, 2013). In this study the author considered difficulties experienced by beginning-level undergraduate students who are beginning to learn the concepts of limit. He used a commognitive framework to analyse the shifts in discourse which the instructor observes as well as the extent to which reification occurs through the change from the use of verbs to that of nouns. The survey he conducted at the end of the study indicated that students considered the limit as more a process than a number. The study attempted to focus on how the instructor's discourse use influenced the students' ability to learn the concept. He found that the instructor primarily used the discourse of discreteness signifying the limit as a number rather than that of a dynamic process. The learners, on the other hand, were predominantly using the dynamic, process-oriented discourse and the author found that this needed to be challenged in order for the required learning to occur.

5.9 Applying the Framework

The aim of this study is to analyse, using the commognitive framework, classroom sessions to determine how teaching and learning occurs and to determine the factors enabling this. These factors could be related to learner activities both individually and in groups or could be a result of teacher initiated and facilitated interactions or a combination of both.

I analyse the transcripts and videos to determine the use of visual mediators and attempt to analyse the efficacy of their use in the discourse at hand. Algebraic artefacts are examined when used to discuss manipulations associated with attempts to gain a greater understanding of infinitely recurring decimals and their relationship to rational numbers. Any use of iconic visual mediators to illustrate the process being followed in the limit associated with an infinitely recurring decimal are assessed to determine the extent to which they provide any sort of "bridge" from the process (which is infinite and unending) to the realisation of the mathematical object which is the actual limit.

I attempt to determine whether any routines used during the classroom session are performed with insightfulness or whether they are merely rituals being used in a "by rote" fashion in order to determine a result. I try to determine whether peer or teacher approval or acceptance is being sought, whether constant feedback or input is required or sought after and whether there is any flexibility in the use of the procedure. This allows me to ascertain the presence or otherwise of learning and the degree of understanding of the mathematical object under discussion.

I attempt to determine if meta-level learning occurs by analysing whether any discourse breakdown occurs as a result of incommensurable discourse and if these discourses are then altered and resolved with or without the assistance of the teacher.

Reification has occurred if the learners are able to work conceptually with the recurring decimal as a resultant limit (being the rational number) without considering the infinite process involved in this limit.

As part of assessing the extent to which successful learning took place, I looked at the extent to which a *learning-teaching agreement* is forged, either implicitly or explicitly between the class and the teacher. The creation of this form of learning contract is a necessary condition for learning to occur (Sfard, 2008, p. 282). In this situation the “old-timer” adopts or is given the role of assisting the “new-comer” to gain entry into the community of discourse (Lave & Wenger, 1991). The teacher usually assumes the role of the ‘old-timer' but this is not necessarily the case, especially when class members are given the chance to lead the discussion i.e. to justify and explain their particular personal discourse idiosyncrasies when attempting to resolve a commognitive conflict. This situation is closely linked to the need to establish a *leading discourse* where agreement is needed between the various parties as to which discourse will be endorsed by the group as a way of resolving the commognitive conflict. This may involve adjustments to each party’s discourse in order to arrive at a final, agreed discourse and endorsed narratives. Implicit in this negotiation are issues of power relationships whereby the teacher traditionally holds the upper hand due to her standing in the community of discourse. She is also usually supported by textbook or other background information which allows her to direct the discussion in ways she has determined.

Thus the learning-teaching agreement requires agreement on the participants' roles, on what is to be accepted as the leading discourse and what the expected change is going to be.

In my Analysis chapter I assess the role played by the teacher in establishing the leading discourse and what learning-teaching agreement is forged. This provides me with a way to determine the effectiveness of the teaching and learning taking place.

6 RESEARCH DESIGN

6.1 Introduction

This study attempts to examine high school learners' ways of dealing with the concept of recurring decimals and their equivalence to a particular rational number. Opportunities for teaching and learning in the context of classroom lessons are analysed to determine the extent to which learners extend their mathematical abilities to manage concepts related to infinity.

The study consists of two 60 minute classroom sessions using the same class and teacher in both sessions. The classroom layout is the conventional form i.e. teacher in front with whiteboard and learners at individual desks which could be clustered if necessary for discussions or group work. The class size was 18 learners about to enter grade 11 at a private co-educational school in the northern suburbs of Johannesburg, close to the University of the Witwatersrand for convenience. The class makeup was predominantly upper middle class with approximately 20% of learners receiving a full bursary for school fees. Hence there was a relatively diverse range of student background. The learners chosen all participate in Advanced Program Mathematics at their school. This class makeup was chosen so that the learners would be relatively mathematically sophisticated and hence more readily able and willing to engage in the interaction with an unfamiliar teacher while at the same time being observed and recorded.

The regular mathematics teacher was not available to lead the lesson so a university colleague experienced in mathematics education and currently a mathematics lecturer for undergraduate teachers in training, presented the classroom lessons.

6.2 Methodology

In deciding on the research methodology to be used, I was guided by the resources at my disposal and the desired scope of the project. I had access to an IEB[2] school which is close to the university and offers Advanced Program mathematics to grades 10, 11 and 12. I therefore felt that one of their grade 11 or 12 classes who are studying Advanced Program mathematics would provide a suitable sized group of respondents which would result in the best chance of usable data. The class I was able to use consisted of 18 learners about to begin grade 11 Advanced Program mathematics.

Cohen et al. (2007) defines this way of acquiring a set of respondents as “convenience sampling” and while acknowledging this as an acceptable method for creating group to study, they make the point that no generalisations to a more general population can be made as a sample selected in this way is not representative of any wider group. I felt this was acceptable for my study which does not aim to make any claims as to applicability or generalisation to any wider population.

The structure I have chosen for the study is to implement my research project as a case study for this research. This strategy allows for “an in depth study of single instance in an enclosed system” (Opie, 2010, p. 74) and is thus perfectly suited to the study of two classroom lessons using the same learners and teacher which forms the basis of my study.

A case study can be one of several types. Yin (1984) describes these as either exploratory, as would be used for a pilot study; descriptive in which case a narrative account will be provided of the events and activities observed; or explanatory in which case theories may be examined or tested. This particular study adopts a narrative approach in which the

[2] Independent Examination Board – an accredited assessment body providing matriculation examinations for private schools in South Africa

recorded data is discussed and analysed in the light of the theoretical framework of commognition.

A case study approach has both advantages and disadvantages. As noted in Cohen et al.(2007) some of the advantages include:

- The presentation is more readily understood by a wider audience, including a non-academic audience.
- The work can be undertaken by a single researcher rather than needing a team
- The work provides insights into similar situations which may improve the applicability of the findings

Disadvantages include:

- Results may not be able to be generalised.
- Results cannot easily be cross-checked and hence may be selective and biased.

For this study I felt that a case study approach was appropriate and that the advantages outweighed the disadvantages in that I am not seeking to make generalised claims related to the findings of the study. The issue of bias will be addressed by using the constructs of the theoretical framework to determine the extent to which learning has or has not occurred and by adopting a non-participatory observer approach.

Cohen et al. (2007) describes the degree of structure in a particular study in two dimensions: One as the degree of structure imposed by the observer ranging from structured to unstructured and the other as the degree of structure in the observational setting ranging from natural to artificial. In this study I would rate the observer imposed structure as unstructured (as I was a non-participatory observer) and the observational structure as slightly more structured than natural as the lessons were created specifically for the study and presented by an unfamiliar teacher.

6.3 Methods of Data Collection

This is a qualitative study designed using an observational research approach as described by (Opie, 2010). I was not able to use interviews as part of the study as I had originally planned owing to a limited time that I had available with the learners who were about to write examinations.. I used the approach of a non-participatory observer. This was done to prevent any reaction caused by the presence of the observer which could influence the study. In addition, this approach prevents the introduction of any of the researcher's perspectives or expectations into the study. It does have the disadvantage that the researcher cannot ask any questions, interrupt or in any other way direct the flow of the lesson. Taking this approach allowed the collection of field notes and observations during the lesson which would not otherwise have been possible.

The benefits of an observational strategy are that it allows the researcher to focus on live and immediate data instead of indirect or inferred data. This provides the potential to "yield more valid or authentic data than would otherwise be the case" (Cohen et al., 2007). In addition recording the data allows repeated analysis of the same data possibly giving rise to additional insights after the event. It also allows consideration of extraneous environmental or other items which may have a bearing on the study. An observational approach also reduces the time commitment needed from the participants when compared to other methods such as interviews (Cohen et al., 2007; Opie, 2010).

In addition to the observations, transcripts and recording, additional data was collected in the form of written worksheets which capture each learner's initial ideas with respect to the core question (is 0.9 recurring equal to 1?) and also which provided some exercises related to this topic. These exercises were intended both as a refresher in the area of rational numbers and decimal representation and also as a challenge to highlight some of the issues associated with calculations using infinitely recurring decimals. Examples of this worksheet and completed worksheets are included in Appendix A.

6.4 Types of Data Collected

In my research I aimed to look at the changes that occurred in learners discourse and how those happened. Thus, although I did not teach the lessons, I co-designed the lessons with the teacher in drawing on work reviewed in the literature review in order to set up situations which could result in commognitive conflict. In order to establish learners' understanding on an individual basis both before and after the lessons, I gave learners a worksheet to complete. The main focus of the data however was the video-taped lessons and learners interactions, both with each other and the teacher in this process.

6.4.1 Video and Audio

The lessons were captured on both audio and videotape for subsequent transcription. The video and audio covered both the classroom lesson as well as the times when the learners worked in small groups on the exercises. No transcriptions were done of these latter videos as the sound quality was poor.

6.4.2 Worksheets

At the start of the first lesson the learners were each given a worksheet with a set of questions covering problems related to rational numbers, fractions and recurring decimals. These exercises culminated in various questions related to decimals including the question: "Is 0.9 recurring equal to 1.0? What justification is there for your answer"? The worksheets were completed individually and were collected. They were not graded or assessed as they were designed merely to have the learners reflect on the subject area and to start the discussion related to infinitely recurring decimals. During the course of the first lesson, an additional worksheet was completed by each learner and collected. This worksheet included further questions on rational and infinitely recurring decimals. These were also collected. At the end of the final lesson a worksheet covering just the major question “Is 0.9 recurring equal to 1.0? What justification is there for your answer” was completed by each learner and collected. These 3 worksheets are included in Appendix A.

6.5 Lesson Structure

The first 60 minute session reviewed the concept of terminating, repeating or recurring decimal representations as well as non-repeating representations (such as π). The learners then worked in their assigned groups to discuss and decide on answers to the various questions. The learners were also asked to provide reasons or justifications for their answers.

The remainder of the first session was used to introduce exercises related to working with various forms of repeating decimals, converting from rational fractions to decimals and vice versa. The exercises used examples similar to those used in Brijlall et al. (2011). These exercises also introduced situations where the use of existing routines related to finite processes is problematic – such as when multiplying a recurring decimal by an integer where a “carry operation” is required or subtracting a repeating decimal where a “borrow operation” is required. These exercises are included in Appendix A.

Some examples illustrating these difficulties are:

1,44444... multiplied by 3 (which is the first 4 to by multiplied by 3?)

1,7 minus 0,88888… (where do you begin the subtraction?)

The second 60 minute session consisted of a lesson given by the mathematics teacher and focused on arriving at the conclusion that 0.9 recurring is equal to 1 through considering various justifications and contradictions which would occur if this were not the case. The various cases discussed in Conradie et al. (2009) and detailed in the literature review section of this report were used with modifications to make them more accessible to the grade 11 learners. These cases range from less to more mathematically rigorous. A summary of the examples used is shown below.

EXAMPLE 1:

Let $x = 0.99999\ldots$

Then $10x = 9.9999\ldots.$

And solving for x results in x=1

EXAMPLE 2:

$0.333\ldots = \frac{1}{3}$

Multiply both sides by 3

EXAMPLE 3:

$\frac{1}{9} = 0.1111\ldots$

$\frac{2}{9} = 0.222\ldots.$

And so on so until finally

$1 = \frac{9}{9} = 0.999\ldots..$

EXAMPLE 4:

$0.9\ldots. = 0.9 + 0.09 + 0.009 + \ldots.$

This is a geometric series. Solve using the formula $S = \frac{a}{1-r}$

The teacher is working towards a conclusion that the overall realisation needs to be that consistency of results is the key desirable feature. In this session the learners work on operations with infinitely recurring decimals which were designed to show that it is necessary to define 0.9 recurring as 1 to achieve consistency of results.

At the conclusion of the second session the class was again split into the same groups as before and asked to reconsider the original questions including “Is 0.9 recurring equal to 1? Yes, no, don’t know/not sure – with justifications or explanations for the answer chosen.” Small group discussions can also facilitate the emergence of misconceptions according to existing research (Gooding & Stacey, 1993). An attempt was made to foster discussion or even arguments between the group members in an attempt to draw out any resilient misconceptions which had survived the two lessons. Sfard (2008) regards misconceptions as errors which occur systematically as a result of the failure to modify portions of a previous discourse which considering a new, revised requirement. Thus the new endorsed narrative contains portions of a modified discourse together with remnants of the previous discourse which still need to be altered.

7 ETHICS, VALIDITY AND RELIABILITY

7.1 Ethics

In accordance with the ethical considerations associated with performing studies involving humans, I applied for and received clearance from the Witwatersrand University ethics committee (clearance provided under protocol number 2011ECE140C). In line with these regulations, all the participants, teachers, the principal of the school and the parents of the learners were invited by letter to participate in the study. These letters were signed and returned and only those learners with signed acceptance forms were included in the study. The learners and their parents/guardians were informed of the purpose of the study, the intention to use video and audio methods of recording and that the learner may withdraw participation at any stage. Confidentiality of both their identity and the school is guaranteed at all times. In addition, the underlying data will be destroyed after 5 years. Pseudonyms are used for both learners and the teacher to preserve anonymity. Samples of the participation requests and acceptance letters are included as Appendix B.

The participants are also guaranteed that there is no link between their participation or non-participation in the study and their scholastic results or personal reputations.

Data storage security requirements are being observed for the data resulting from completion of the data gathering exercise.

The school has requested a copy of the final research report so that they may get some insight and feedback on the standard of the learners in this particular class. No teacher from the school participated so there is no issue related to feedback (positive or negative) related to school staff members. The report will also not be used to assess in any way, the lecturer from Wits who conducted the lessons.

7.2 Validity and Reliability

One definition of Validity is that it is “the degree to which a method, a test or a research tool actually measures what it is supposed to measure” (Wellington, 2000 quoted in Opie, 2010). In a qualitative study such as the current one, the measurements are the degree to which teaching and learning occurs. Using the theoretical framework chosen for this study, one therefore has to verify that a change in discourse which is regarded as evidence of learning is appropriate, especially in the particular context of a mathematical understanding as in this study. I have justified the use of this particular approach by referencing in my theoretical framework chapter several similar studies published in international peer-reviewed journals which have used the same approach. These studies have used similar techniques to those used in my study to determine the extent of teaching and learning in subject areas such as functions, algebra and infinity.

The reliability or credibility of a study is determined by a number of factors including amongst others:

1) Explanation of data gathering procedures
2) Transparent presentation which enables re-analysis
3) Reporting of negative instances
4) Acknowledgement of bias (Sturman, 1999 quoted in Opie, 2010)

I have addressed these issues by preforming a verbatim transcription of the lessons which are provided as transcripts in Appendix B. This is an approach suggested by Sfard (communicated in presentations). The data collection can thus be regarded as reliable. The extent of bias has been reduced as no interviews or marking of papers was performed and in addition, I did not have any influence over the lesson progression once it began.

A limited amount of triangulation is possible by comparing the recorded information from the lesson transcripts with the worksheets. This will serve to increase the credibility of the findings.

8 ANALYSIS AND DISCUSSION

8.1 Introduction

The lessons I analyse consisted of two 60 minute lessons with a class of 18 grade 10 learners who are about to enter grade 11. In this section of the report I assess what transpired in the lesson by reviewing the video recordings, lesson transcripts and worksheets handed in by the learners.

The sections of the transcript that I chose for analysis and discussion are those related to discourse changes; either where the focus was on a clash between two or more discourses or where a discourse was being developed or extended. This was done by analysing the transcripts (and the videos where required) of the lesson and searching for interactions between class members and the teacher where different word usage was evident, or an altered or new routine was being introduced.

An example of this occurs at minute 43 of Lesson 1. At this point in the lesson the teacher is summarising and interrogating 3 groups in the class who have arrived at different methods and final results when adding two infinitely recurring decimals. She says "So one group said, look I am just going to write the sum like this. I go four plus seven is one, carry the one so I get one point one and of course the bar stays right? So its recurring". This is a new routine giving the (erroneous but plausible) result $0,\bar{4}+0,\bar{7}=1,\bar{1}$. In examining the video one can see that she labels this "number one". Next the teacher attempts to explain the routine developed by another group. "and you go four plus seven is one and carrying on the one and then four plus seven is eleven plus the one from the previous is twelve and that is going to keep on going right? So I will get…so what I have got is sort of a funny thing with a whole lot of two's, infinitely many two's and a one at the end right? So this (1,222…21) is what they mean over here infinity many twos right?" In examining the video one can see she writes the following on the board:

$$\begin{array}{l} 0,444...44 \\ 0,777...77 \\ \hline 1,222...21 \end{array}$$

Thirdly, she describes the last group's routine as follows: "The back table said it is not going to be like that (pointed at 1,222…22) it is going to be one comma two recurring (Written on the board and labelled as number three) because this (as meaning the infinity of twos) is going to go on so it is keep going to be one point two recurring because this is just going to go on, right?"

I interpret this interaction from minute 43 to minute 45 of lesson 1 as showing evidence of 3 incommensurable discourses which gave rise to a commognitive conflict and hence I selected this episode as part of my analysis.

8.1.1 Lesson Plan Design

The lessons were structured in an attempt use the learners' prior knowledge of rational numbers, sequences and series and to extend this knowledge in order to examine certain aspects of infinity.

I designed the lesson plan in conjunction with the teacher who was going to conduct the contact sessions with the class. This was not the learners' regular teacher, but instead a mathematics education specialist familiar with the theoretical framework being employed. The rationale behind the lesson plan design was to employ the hopefully familiar finite discourse of rational number arithmetic and to move to a discourse of infinite arithmetic through introducing infinitely recurring decimals by linking them to the rational number for which they are an alternate representation. This was aimed at utilising their existing knowledge of certain rational numbers as infinitely recurring decimals but with the aim of introducing commognitive conflict by means of carefully chosen examples which would hopefully give rise to incommensurable discourses. The act of resolving these incommensurable discourses will facilitate meta-level learning. As a result of these changes, words will be used in a different way and with new meaning, resulting in greater mathematical sophistication in the participants in the discourse. The exercises and examples which the class were given to work on in groups were all chosen with this aim. The various worksheets that the learners completed are provided in Appendix A.

Once the lessons began I had no further influence on their trajectory and merely partook as an observer and recorder.

8.1.2 Initial Question

In order to get an overall picture of the individual learners' understanding of infinitely recurring decimals before and after the lessons, each learner was asked to answer and to justify their answer to the following question:

"Is $0,\bar{9} = 1$? Explain or justify your answer".

This question formed part of the worksheet the learners were asked to complete at the beginning of the first lesson and again at the end of the final lesson.

A review of the responses provides an assessment of each learner's prior knowledge of this area as well as any possible changes in their views after the classes; changes which potentially result from the teacher's contribution, the discussions held during the class as well as their own reflections and activities. As these lessons were conducted within days of each other, it is unlikely that any other outside influences were involved.

8.2 Analysis

Analysing the responses of the 18 learners who made up the class shows that before the classes, 9 learners answered “No” and after the final lesson 7 learners answered “No”. Of the 9 learners who had initially answered “No”, 3 changed their final answer to “Yes”. In addition 5 learners answered “Yes” before the classes and 7 answered “Yes” after the classes. 2 learners who had answered “Yes” at the start changed to answer “No” at the end.

All of these learners provided some form of justification or explanation for answering the way they did. The responses of learners who did not answer or answered with a version of “unsure” were excluded from these totals. These responses are summarised below.

Table 8.1 Counts of learner responses

Initial answer “YES”	**Initial answer “NO”**	**Changed from :NO” to “YES”**	**Changed from “YES” to “NO”**	**Final answer “NO”**	**Final answer “YES”**
5	9	3	2	7	7

The results shown in Table 8.1 indicate that half the class of 18 learners held the incorrect view at the start of the lessons and this situation had not changed significantly by the end of the second lesson, although within these results some individuals had changed their viewpoint.

In reviewing the justifications for those who answered “No” to the above question at the start of the first lesson, it is apparent that the learners are using a discourse related to the finite situation in attempting to analyse the situation of an infinitely recurring decimal (which is in this case a limit although that fact is not expressed to them). They consistently use terminology that refers to a concrete difference resulting in $0,\bar{9} < 1$.

For example:

> "It is the closest number to but less than 1 no matter how close it is it is still that infinitely small decimal smaller".
> "No, because there will still be that very small, small, extremely small difference".
> "No, because there will always be that little bit missing".

These learners are apparently using the visual mediator which is the representation of the number containing a decimal fraction which is apparently smaller than 1 because it begins with the leading zero of $0,\bar{9}$. They are constructing a finite approximation of the infinitely recurring decimal fraction and hence arrive at the given conclusion. This concept of a limit as being an approximation and hence never actually attained is described in (Gucler, 2013, p. 447) as one of the ways learners describe a limit.

Other learners who answered "No", show a lack of reification of the concept of limit which is necessary for analysing an infinitely recurring decimal fraction. They use verbs describing a process rather than nouns describing an object. In expressing it in this way, they also use an approximation view and end up with a finite difference of 0,1 between $0,\bar{9}$ and 1.

For example:

> "No, because the 9 will go on forever so you won't reach the target which is 1. There will always be that 0,1 difference".
> "No, it is not equal to one because even though it is the closest number to one, it will forever recur and cannot be rounded to any decimal place".

These learners are using the endorsed finite narrative that 0,9<1 to infer that $0,\bar{9} < 1$.

I review the learners' responses and justifications of their answers to the same question at the end of this chapter.

In order to probe the interaction in the classroom and the way it either caused (or did not cause) learning to occur I conducted a detailed analysis of the lessons using the video recordings and written transcripts.

In accordance with the theoretical framework being used, the presence or absence of learning will be assessed by the changes, or lack thereof, in the mathematical discourses being used. For the purposes of this analysis, learning mathematics is *defined* to be a change in discourse (Sfard, 2008, p. 255). So a key part of my analysis is to determine if the learners are able to partake in a changed discourse which arises during the course of the lessons in order to deal with the issues raised by working with infinitely recurring decimals rather than finitely terminating decimals.

8.2.1 Lesson One – the beginning of conflict

The teacher begins the lesson by explaining that the class is going to look at some of the ideas related to infinity and that "it is quite mystical". She explains that people struggled for centuries with these ideas and that some of the issues which surface are "uncomfortable and don't make sense". She introduces a modified version of Zeno's paradox using an infinite process of halving the distance to be travelled to a particular point, thereby never arriving at one's destination. She contrasts this with the learners' experience of the "real" world, in which it is clearly possible to travel a particular distance in a finite length of time.

Introducing this example raises a number of issues designed to disturb the learners' equilibrium:

1. It introduces in a non-rigorous fashion the concept of potential infinity represented by an infinite process which results in a goal (limit) which appears to be impossible to attain, but which is actually achievable when considering the learners' experience of the physical world.
2. It creates a disjunction in the mind of the learner between the mathematical concept being discussed and its result as opposed to the result provided in the physical world. This example creates a *cognitive* conflict for the learner which is usually resolved by some rational reflection on the part of the learner. This approach is characterised by its acquisitive nature as opposed to a commognitive conflict which occurs though incommensurable discourses and has to be resolved by the learner through adoption and individualisation of the new discourse, usually with the assistance of a teacher or mentor. This second approach, in contrast, is characterised by its participative nature (Sfard, 2007).
3. It alerts the learners that prior naïve knowledge and experience may not prove sufficient to operate in this extended mathematical context and that the finite endorsed narratives may not be applicable to the world of the infinite.

In fact, to reinforce what has transpired, the teacher announces "And this is what we are going to be messing around with your mind, right"?

The teacher then splits the class into groups in order to do the first worksheet. These problems are simple exercises involving rational fractions, conversion to and from decimal fractions and some conversions of simple and well-known recurring fractions such as $0,\bar{3}$. This is to remind the class about the endorsed narratives related to rational arithmetic which they should already know. Included in this worksheet is the question described at the beginning of the analysis section. This worksheet forms part of Appendix A.

To consolidate the work introduced in the worksheets (see Appendix A), the teacher reviews the words being introduced such as Rational Fraction, Denominator, and Integer etc. In addition, she discusses the visual mediators, narratives and definition used to describe and manipulate these mathematical objects. For example, she explains that:

> "A Rational number is a number which can be written as:
>
> $\frac{A}{B}$ where A and B are integers and B ≠0."

She also challenges these definitions by exploring the boundaries and asking questions such as "Is $\frac{\pi}{3}$ a rational fraction?" In doing so she is showing the importance of using precisely defined mathematical terms in order to create the discourse being used in the lesson. She is also beginning to forge the Learning-Teaching agreement which is a necessary condition for learning to take place (Sfard, 2007). She may use this agreement to create a leading discourse over the course of the two lessons.

The endorsed narrative that a rational number is either a terminating decimal or else a repeating decimal with some pattern recurring to the right of the decimal point is then established. So far no opportunity for new teaching and learning occurs as the class is comfortable with the concepts and narratives being discussed. The teacher introduces the visual mediator related to an infinitely recurring decimal faction, namely the usual representation of a decimal fraction but with a bar or dot above the pattern that repeats infinitely many times e.g. $0,2754\overline{293}$ or $0,2754\dot{2}\dot{9}\dot{3}$. She explains this colloquially as:

"Is just the same thing, there is nothing fancy and here what we would do , all things that appear that aren't repeating, we just put them in here and then if we put the bar over all of those it means from this point on, you just repeat 293 293 293".

She is creating here a specific realization for the object Rational Number. Another realisation of the same Rational Number would be $\frac{2751539}{9990000}$ as can be verified with a calculator. The difficulty here is that there is no obvious visual similarity between these two representations. This is present already in the learners' current mathematical context e.g. $\frac{1}{4}$ and 0,25 have no visual similarity either. They are, however, comfortable and familiar with this latter example and can always verify the equivalence of these by performing simple division of numerator by denominator or by representing 0,25 as a fraction and then simplifying. This is not easily possible in the case of infinitely recurring decimals.

8.2.2 Interaction 1

The first presentation in front of the class is by a learner to show the conversion of $0,\bar{3}$ to a rational number. The consensus in the class in prior discussions was that this was "known" to be $\frac{1}{3}$ based on prior teaching. See Table 8.2 for details of this example.

Table 8.2 Converting a basic rational number

Time (mins)	**Transcript / Action**	**Mediator**	**Words**	**Routines and Narratives**
23:00	*[Learner writes Let X=0,*$\dot{3}$ *10X = 3,* $\dot{3}$ *10X-X = 3,*$\dot{3}$ *– 0,* $\dot{3}$ $\frac{9X}{9} = \frac{3}{9}$ *X =* $\frac{1}{3}$*]*	Symbolic: written equations.	None.	Ritual.

In this interaction, there are noticeably no words involved. The learner works quickly, without any hesitation and with no pauses between the various steps she writes on the whiteboard. There is total silence in the classroom and she makes no attempt to explain or justify her workings. After completing the write-up on the whiteboard she turns towards the teacher and the rest of the class as if looking for acknowledgement which is given by means of applause. She appears pleased, smiles and returns to her seat.

Although the learner demonstrates fluency in arriving at a correct answer, she shows no obvious evidence of any understanding whatsoever of the mathematical artefacts (conceptual or procedural) being used. This may be an instance of the use of *ritual* in order to complete a task. This would be the case if the overriding aim is one of social acceptance or recognition rather than completion of the task at hand. The learner provides no justification that the routine being used is applicable in this situation, nor does she explain any reasoning behind the steps. This is a technique that the class has learned previously and so the insight she shows is to recognise that the approach taken fits the problem at hand. The learner uses the familiar approach for solving an equation in one unknown but there is no discussion as to whether such an approach can be extended from the finite to the infinite. She seeks positive feedback from the class and teacher by turning to them and smiling when she completes the activity, so there is a strong suggestion that this may be an instance of the use of ritual (although she was not specifically asked to explain or justify the approach taken). This interaction is reminiscent of the performance of a clever “party trick” by the learner.

The opportunities for learning afforded by the teacher in this instance can be assessed by examining the subsequent interaction from the transcript. See Table 8.3 for details of this interaction.

Table 8.3 Converting a rational number with period 2

Time (mins)	Transcript	Mediator	Words	Routines and Narratives	Comment
23:00	*[Teacher speaking]* Right, anyone see what she did in the method? And how did this one differ then? *[T points to* $0,\overline{54}$ *on the board]*				Routine being used is the same as used by the learner previously.
	[Teacher writes] $0,\overline{54} = X$ $100X = 54,\overline{54}$ $99X = 54$ $X = 54/99$	Symbolic: Equation representing recurring decimal manipulations.			Teacher has written equations of solution on the board.
	[Teacher speaking] Instead of multiplying X by 10 multiply it by 100.		Describing routine being used to solve equation.	Teacher uses routines from endorsed narratives of equality and equation solving.	

In this case the teacher extends the routine to include recurring decimals with a period greater than 1. She does this by working through a different example using a recurring decimal with period 2. She does this without any conceptual explanation and assumes that the entire class is familiar with this routine. In this instance the teacher does not explicitly afford the class any opportunity for new learning as she assumes (with agreement from the learners) that they are all familiar with this particular routine for converting a recurring decimal of arbitrary period length to the corresponding rational number. The teacher confirms with the class that she is using a routine which constitutes an endorsed narrative from the viewpoint of the class by saying at minute 23 " Ok so I just wanted to check that we all had that method somewhere in our heads, that it was probably taught earlier, I am hoping".

8.2.3 Interaction 2

The teacher next starts reviewing a discussion that each group has had about the final question in the problems the learners were asked to do. One of the learners applies the routine related to equality of both sides of an equation to show that $0,\bar{9}=1$.

He does this as follows:

$$\frac{1}{3}=0,\bar{3}$$
$$So$$
$$3\times\frac{1}{3}=3\times 0,\bar{3}=0,\bar{9}$$
$$So$$
$$1=0,\bar{9}$$

This again assumes that the routine of finite multiplication can be extended unchanged to infinitely recurring decimals. The teacher does not comment or offer any feedback other than to point out that others at his table did not agree with him.

One of the other learners L1 objects to this result and tries to link the discussion back to the process of taking steps which are half as long as the previous step as used in Zeno's paradox. "No matter how many halves you take, there is always a tiny bit away from it..". L1 is immersed in the discourse of the finite and must still make the transition to the infinite in order to be able to attain the limit rather than approximating it with a finite result.

At this point, the teacher is allowing commognitive conflict to occur by having the learners develop incommensurable discourses which are giving rise to different results. At this time, the levels of frustration and despair are rising amongst the learners as evidenced by nervous laughter and giggling.

8.2.4 Interaction 3

In the next interaction the teacher reviews certain problems that the learners have been working on in their respective groups. The teacher is now trying to create additional commognitive conflict by challenging and disrupting the assumption from interaction 1 that the routines of the finite can be translated without change to the infinite. This she achieves by reviewing a problem involving addition of infinitely recurring decimals involving the notion of "carrying". The problem the class is presented is the following:

$$\begin{array}{r} 0,\bar{4} \\ + \quad \underline{0,\bar{7}} \end{array}$$

In this case, 3 groups in the class get different solutions, all of which they are able to justify as being "reasonable".
The solutions they get are as follows:

Solution 1	**Solution 2**	**Solution 3**
$\begin{array}{r} 0,\bar{4} \\ + \quad \underline{0,\bar{7}} \\ 1,\bar{1} \end{array}$	$\begin{array}{r} 0,444...44 \\ + \quad \underline{0,777...77} \\ 1,222...21 \end{array}$	$\begin{array}{r} 0,\bar{4} \\ + \quad \underline{0,\bar{7}} \\ 1,\bar{2} \end{array}$

This is a clear indication of incommensurable discourses arising through the extension of the routines of finite arithmetic to the case of the infinite. See Table 8.4 below for details and analysis of this interaction.

Table 8.4 Adding recurring decimals

Time (mins)	Transcript / Actions	Mediator	Routines and Narratives	Comment
43:00	*[Teacher speaking]* So one group said, look I am just going to write the sum like this. I go four plus seven is one, carry the one so and I get one point one and of course the bar stays right? So it's recurring. *[Teacher: writes on the board and labels it as number one,* $0{,}\bar{4}$ $+ \quad \underline{0{,}\bar{7}}$ $\underline{1{,}\bar{1}}$ *]*	Symbolic: representation of calculation.	Standard routine from endorsed narrative of addition in finite case.	Teacher creating commognitive conflict by extending discourse to the infinite.
	Then another group said ok, alright let's write it out and you go four plus seven is one and carrying the one and then four plus seven is eleven plus the one from the previous is twelve and that is going to keep on going right? So I will get…so what I have got is sort of a funny thing with a whole lot of two's, infinitely many two's and a one at the end right? So this (1,222**…**21) is what they mean over here infinity many two's right? *[Teacher writes on the board and labels it as number two,* *0,444…44* + *<u>0,777…77</u>* *<u>1,222…21</u>* *]*	Symbolic: representation of calculation.	A variation on routine from finite arithmetic.	Teacher extending the discourse to the infinite again in a different way.
	The back table said it is not going to be like that (pointing at 1,222…22) it is going to be one comma two recurring because this (the infinity of two's) is going to go on so it is keep going to be one point two recurring because this is just going to go on.*[Teacher writes and labels as three,* $0{,}\bar{4}$ $+ \quad \underline{0{,}\bar{7}}$ $\underline{1{,}\bar{2}}$ *]*			A third possible extension to the discourse of finite arithmetic.

What is happening at this point in the lesson is that the teacher is creating commognitive conflict by allowing the development of 3 distinct discourses as detailed above to deal with the problem of adding two infinitely recurring decimals which involve carrying. She allows the class to discuss and challenge the various solutions without resolving the situation. At minute 47 she states "Ok, these are interesting points and I am deliberately not judging at this point because what we are doing is bringing out some of the interesting debates which are important for all of us to think about it." She makes it clear to the class that it is impossible to automatically extend the discourse from the finite case to the infinite. She thus creates the opportunity for learning by explicitly developing this conflict and also allows time for the learners to analyse and grapple with the inconsistencies themselves instead of instantly resolving them. We now have 3 potential narratives, none of which is endorsed at this time. In fact, the teacher refuses to take the lead at this time, and defers further resolution till the next lesson.

Another group decides to side step the issue entirely, in minute 45 ML2 says "We turned it into a fraction so it have the same denominator ….." The teacher acknowledges that this is an approach that provides a workable solution but asserts her power in the relationship to force them to return to working with the decimal representation. This could have been an opportunity for learning by explaining how in the field of mathematics one often uses isomorphic transformations to move a problem from one area to another where there are better tools and then transforms the solution back to the original area. Exploring this approach would have been an opportunity to engage in meta-level discourse by exploring the various object level discourses used in each area of mathematics. The idea in colloquial terms of working smart instead of hard would have been applicable here as the arithmetic issues encountered with extending the discourse from finite to infinite are avoided by working in the area of rational fractions. She did not pursue that opportunity.

8.2.5 Lesson Two

In the subsequent lesson the teacher presents the 3 arguments from the previous lesson which attempted to justify that $0,\bar{9}=1$. These are presented in the first 4 minutes of the transcript and are summarised below:

1) The algebraic use of an equation and appropriate manipulation to solve for the unknown which is the recurring decimal fraction.
2) An argument using the endorsed narrative that $\frac{1}{3}=0,\bar{3}$ and then multiplying both sides of the equation by 3 giving $1=0,\bar{9}$.
3) An extension of the pattern that $\frac{1}{9}=0,\bar{1}$ and $\frac{2}{9}=0,\bar{2}$ etc. ending with $1=\frac{9}{9}=0,\bar{9}$.

These are referred to as Argument 1, 2 and 3 in the transcript and subsequent discussion. She allows the class to reflect on these 3 arguments, then asks for comment and facilitates a discussion amongst the learners. See Table 8.5 below for excerpts from this interaction.

Table 8.5 Alternative arguments

Time (mins) / Speaker	Transcript / Actions	Mediator	Routines and Narratives	Comment
4:21 T	Are any of those arguments convincing to you and do you have problems with any of those arguments? Right. So let's just… let me give you 2 seconds. Lean to the person next to you or behind you and just talk to the person next to or behind you. Any of these arguments or all of them do they convince you 0,99 recurring is 1 or is there something in these arguments that worries you? Talk to someone for a couple of seconds and then we will… *[learners discuss this]*	Verbal: explanation of the workings of the calculations.	Endorsed narratives from algebra of rational fractions.	Teacher is using existing narratives to try to justify her conclusion that 0,9 recurring = 1.
6:20 FL1	In argument 3 it says 0,99 recurring equals 1 so it's kind of like rounding off, but does 0,888 recurring equal 0,9 because 0,9 is in a fraction like represented as 9 over 10. And 9 over 10 does not equal 8 over 9.	Algebraic: representation of finite length decimal.	Endorsed narrative of rounding to a particular number of decimal places.	Learner is attempting to equate an infinite process with single action and resulting object, in this case a rounded result. Start of an attempt to "make meaning" of the previously intractable recurring decimal.
07:07 ML1	Just like, sort of like an argument we did last week was if… Ok, so 0,9 plus 0,1 that's 1. 0,99 plus 0,01 is 1. So is 0,9 to infinity plus 0,000 with a 1 where you see it continuously into infinity, would that equal 1 or would that equal 1,00001 proceeding to infinity?	Algebraic: New learner-created representation of decimal.	Unendorsed narrative proposed by learner.	Introducing an infinite number of 0's followed by a 1 to try to remove the "tiny gap" between $0.\overline{9}$ and 1 discussed in previous lesson

Time (mins) / Speaker	Transcript / Actions	Mediator	Routines and Narratives	Comment
08:06 ML1	Like you look at this and it all makes sense, but at the end of the day you're looking at 2 different numbers that shouldn't be equal to each other.	Algebraic: two different representations of decimal numbers.		Learner rejects equating numbers which have different decimal forms.

In the interaction above, the learners are grappling with agreeing that these different representations are actually equivalent. They are not able yet to adopt the process of *saming* whereby "a certain closed subset of endorsed narratives about one of these objects is isomorphic to a certain closed subset of endorsed narratives about the other object" (Sfard, 2008, p. 170). They are disturbed by the fact that the decimal representations are different and conclude therefore that the numbers cannot be the same. FL1 is also attempting unsuccessfully to use a discourse related to rounding (which is a finite process) in the case of the infinite decimal.

The teacher then re-introduces the issues which were highlighted at the end of the first lesson as a result of the 3 conflicting discourses which were presented by the class to address an aspect of arithmetic related to infinitely recurring decimals, namely dealing with "carrying" when adding two infinitely recurring decimal fractions. She explains how these arguments all rely on the assumption that addition and multiplication work in the same way as for non-recurring decimals and she then refers to the examples from the end of the previous lesson (see Table 8.4) which showed that this assumption is not tenable as it results in different and inconsistent results. In that interaction, the class ended up with 3 distinct, incommensurable discourses from 3 different routines which illustrated the breakdown of a verbatim extension of the discourse of finite arithmetic to infinite recurring decimals. Previous examples had avoided this issue as they were carefully chosen to prevent this eventuality.

In reminding the learners of this situation, the teacher is creating the opportunity for the learners to reflect on their previous assumptions related to this algorithm in a critical light. This reflection on the multiple discourses is an opportunity for meta-level learning through extending the discourse to include these additional situations. She is also establishing herself as more experienced knowledgeable participant and hence is developing the learning-teaching agreement whereby she is going to establish the leading discourse by resolving the issues the class is currently grappling with. Refer to Table 8.6 below for an excerpt from this interaction.

Table 8.6 Multiplying recurring decimals

Time (mins) / Speaker	**Transcript / Actions**	**Mediator**	**Routines and Narratives**	**Comment**
11:03 T	The minute we started playing with these infinite decimals it was fine for a while but then when we… so it was fine doing things like multiplying by 10 and multiplying this thing here by 3. *[T points to the example on the board.]* But remember when we tried to multiply this thing here by 4? If we tried to go 0,33 recurring multiplied by 4: 0,333 x 4 if we tried to do it in the decimal form we got ourselves into a little bit of trouble, right, because we couldn't quite figure where the right place was to start. Is it 4 times 3 is 12 carry 1, then 4 times 3 is 12 plus 1 is 13 carry 1, right? Do we end up with 1.333 *[T writes: 1,33.3…]* with a 2 somewhere at the end or is it just 1,333 recurring, right?	Symbolic: representation of calculation. Visual: written form of decimal fraction	Endorsed narrative of addition from non-recurring decimal arithmetic with standard routine of carrying.	Teacher is implicitly extending the routine which is endorsed in finite arithmetic to the new domain of infinite arithmetic.

At this point at least one of the learners is developing and using the language associated with limits and infinite expansions. For example, ML1 says at minute 15:18 the following in connection with the decimal expansion of $\frac{1}{3}$. "Well it's the closest thing we can write in our ten numeral system that can get to a, to a third. But it's not, it's not that, that's why it continues into infinity. It's infinitely closer but never reaching it though. So say we had… If we had a system instead of like 10's (so our whole number system works in 10's) if we worked in 9's it would be much simpler because we could have nice representation but then 2 would be and a half would continue like this to infinity as well."

This comment indicates an understanding that the issue of recurring decimals is not as a result of the particular base used for number representation but will be present for some numbers in any representation, regardless of the base chosen for the representation. ML1 is showing an understanding of the fact that this issue exists independently of representation and is therefore providing some evidence that learning has occurred, even if it is peripheral to the core question related to $0,\bar{9}$.

8.2.6 The Need for Consistency in Mathematics

The teacher then begins an explanation of the reason why consistency of results is vital in mathematics. The example she uses is the familiar $\frac{1}{3} = 0,\bar{3}$ scenario. The rationale used is that the meaning of the rational fraction is "1 divided by 3" and that performing long-division with this results in the recurring decimal $0,\bar{3}$. This fraction and the recurring decimal must therefore be defined as the same, otherwise inconsistent results will be obtained. The endorsed narrative is that of long-division which divides denominator into numerator to get decimal expansion.

The teacher then begins an explanation aimed at differentiating between a finite representation and an infinite by using $0,\bar{3}$. She extends the example of getting “very close” to $\frac{1}{3}$ by using a large finite number of 3’s in the decimal expansion and describes how this will always be slightly less than $\frac{1}{3}$, no matter how many 3’s are taken for the expansion (provided this is a finite number).

At minute 20:27 the teacher explains:

“So, just imagine that I spent enough time here that I could fill the entire board with 0 comma and then 3’s, right? All the way to the end here. *[T points to the right hand bottom corner of the board.]* I’ve now got like two thousand 3’s, right? I’d agree with you I’ve got very close to a third, but I haven’t actually got there. So let me ask you. Would you agree with me that that’s where I’ve got? So imagine I’ve managed to keep on filling this board all the way to the end here with 3’s, have I got quite close to a third but not actually got there yet? The routine the teacher is using is that of a finite process which achieves a result ever closer to 0,3 recurring while always remaining less than 0,3 recurring.”

And at minute 22:03 she states: “So one has to get in your head, and this is the really difficult thing with infinity, there is a difference between a very, very, very, very, very, very, very, very, very big number and infinity. There is a qualitative difference. And so, when I put those three dots there *[T points to the 3 dots after the* 0,333 *that she has already circled]* I cannot just be thinking of a very, very, very, very, very, very big number. I have to be thinking of a number that goes on forever. And a number that goes on forever is different and behaves differently to a number that actually stops – even if it stops after a long time.”

What she is establishing is that there is a fundamental difference in the discourses associated with a terminating decimal and that of an infinitely recurring decimal. So there is a deficiency in applying our existing discourse from terminating decimals as seen in the breakdown of certain arithmetic operations. This must be rectified in order to be able to include infinitely recurring decimals into our repertoire of routines and hence form part of our endorsed narratives.

So, knowing or at least agreeing that $\frac{1}{3} = 0,\bar{3}$ from the previous discussion, it is necessary to find a result for other recurring decimals which is consistent with this result. Specifically we need to find a result which can then be taken as the definition of $0,\bar{9}$. The teacher uses $0,\bar{3}$ as the basis for this subsequent discussion as the class is in agreement that it is an alternative representation of $\frac{1}{3}$. The teacher is transferring the use of the endorsed narrative for $0,\bar{3}$ to the case of $0,\bar{9}$.

In the transcript at minute 31:05 she states:
"Ok, so our job at the moment is to give a meaning to 0,333 recurring. That is what we are going to do. And I know this is very high level, so just see whether you can keep up with it as we go and then we'll give you a bit of a chance to work with it. *[T writes: 0,333…]* One way to think of what 0,333 recurring is, is that what you've got is a whole lot of decimals that just go on, right? You've got… First of all you've got the 0,3. Then you add to it… *[after 0,333… T writes = 0,3 + 0,03]* the second place. Yes? Then you add to it the third place. *[T adds on + 0,003]* Ja? And then we keep on doing that for ever and ever. *[T adds on +]* Does that feel ok? So that seems like one step in our definition that we can just say it's adding all these little things together."

She continues building this discourse by using finite partial sums to establish the required representation of $\frac{1}{3}$ to an arbitrary precision. She convinces the class that this representation, wherever it is terminated finitely, will be less than $\frac{1}{3}$. At minute 41:08 she explains:

“You’re saying you’ve got one third and you’re subtracting off that tiny little fraction. So any of these 0,3333 things, as long as you’ve got a finite number of 3’s is just a teeny bit little (let me try and get my teeth around this) a teeny little bit less than a third. Do you agree with me?” To which some of the learners reply in the affirmative.

She continues with an explanation of how the sequence of partial sums is constructed. In minute 42:39 she explains “So what we have is we have *[T cleans part of the board]* a whole sequence of little things like this; 0,3 which is smaller than 0,33 which is smaller than 0,333 which is smaller than 0,3333 etc. etc. All of them being smaller than a third.” *[T writes: 0,3 < 0,33 < 0,333 < 0,3333 < ⅓].* At this point she is using a visual mediator, namely the algebraic representation of the string of inequalities to attempt to represent the infinite process of constructing $0,\bar{3}$. The class is completely silent at this time.

She then attempts to reify the concept of limit which is to create a noun, i.e. the number which is the limit, out of the infinite process described above. She attempts this in minute 43:19 by defining what is meant by taking n, the number of terms in the sequence, to be infinitely large: “So what we want is we want to make a definition. We are going to say what it means when we take n being infinitely large. And what is the only option we can take? What is the only number we can give that will work here? The only thing that we can say that will work is we say that when n gets infinitely large this thing has to be equal to one third. And why is it? Why do I say the only possibility we can have is that it must be equal to a third? Because if I look here *[T points to:* $S_n = 1/3\ (1 - 1 / 10^n) = 0,333$*]* it can’t get bigger than a third, right? Because for each one of these little things I have just shown it’s one third minus a little less. So for every single one of the finite ones it’s a third minus a little piece.”

At minute 45:40 ML1 asks a critical question: “But so like you’re saying it’s infinitely close so it must be the same?” This question indicates that he may be at the point of being able to reify the concept of limit as he is describing the option of changing the process into an object (the number $\frac{1}{3}$, The teacher confirms the accuracy of this statement but instead of accepting it, ML1 goes on to argue that there is still some difference between the infinitely recurring decimal and $\frac{1}{3}$. At minute 46:19 he states: “So surely the fact that it’s infinitely close you can also argue that it’s also an infinitely small difference, but the difference is still there”. This indicates that he has not succeeded in reification and has not moved from the concept of potential infinity (i.e. the infinite process) to attained infinity (the limiting value of $\frac{1}{3}$).

The discussion continues between the teacher and ML1 in trying to resolve the issue of a small difference but the learner remains unconvinced. He ends at time 50:40 with the comment: “Saying it’s the biggest pins it down which makes it finite. Does that make sense?” He is unable to reconcile the infinite process with the finite limit at this time although he understands the steps of the argument presented by the teacher and contributes the most feedback when compared to the rest of the class.

The teacher proceeds by repeating the same argument of the infinite sequence of partial sums but using 0,9 instead of 0,3 in order to arrive at a similar limit, in this case that $0,\bar{9}=1$. She states in minute 54:09 the following: “What do we define it to be when n is infinitely large? We say, same as we did there, when n is infinitely large it must be that number *[T points to 1]* that these things are approaching. *[T points to 0,9 < 0,99 < 0,999 < 0,9999]* So we define 0,999 recurring to be equal to 1. And that’s why it is. All right?” At this point the teacher has attempted to create the leading discourse of arriving at a consistent result for converting recurring decimals into the rational number which is the limit. She has done this without using the arithmetic of the finite which was shown previously to have difficulties when extended to the case of infinite recurring decimals.

The class then proceeds to complete the final worksheet (see Appendix A) which asks the same question as before, i.e. “Is $0,\bar{9}=1?$ Please justify your answer.” This question is designed to see if any changes have occurred in the learners’ viewpoints which would indicate whether any learning has occurred.

8.3 Discussion

The research questions which were posed in the initial chapter are repeated here for convenience:

1. What changes of discourse occur during or between the lessons in the learners’ dealings with infinity?

There is a hint of a potential discourse change in lesson 1 when one of the learners attempts to formulate a discussion related to an alternate representation of an infinitely recurring decimal (lesson 1 minute 29). In this instance he describes a situation with infinitely many zero followed by a final 1. This is represented thus: 0,0000….0001 where “….” represents infinitely many zeroes. He proposes this approach for dealing with the infinitesimal amount missing from $0,\bar{9}$ to make it equal 1. This change does not endure as another learner argues later when this representation is discussed again (lesson 2 minute 8:09) that having a final 1 somewhere means the representation is essentially finite. If fact, this latter learner is close to appreciating the discourse change necessary to attain the limit.

During lesson 1 the class breaks into groups which develop three distinct and incommensurable discourses related to adding infinitely recurring decimals involving “carrying”. This is never explored further so there is thus no final extended discourse of decimal addition developed to include this situation.

At the end of lesson 2 the teacher does establish a changed discourse to extend the existing one of arbitrarily long but finite decimal expansion to include decimals with an infinitely recurring pattern. As described previously, the class accepts the arguments leading to this change but does not unreservedly adopt the new discourse. This is evidenced by the predominant tendency not to accept that 0,9 recurring is equal to 1. See Table 8.1.

2. How do these changes occur?

The discourse changes occur in two ways. Firstly as a result of the learners' group work when they discuss a challenging problem and report back to the class as a whole. These changes are generally transitory as they have not been adequately explored and are thus not robust enough to withstand the scrutiny of the other learners. The learners do, however, show a willingness and interest in learning how to craft mathematical items such as individual and unique routines and, because the class and teacher are supportive and not judgemental, they are discussed in a respectful way.

The other way in which the changes occur are through the actions of the teacher. In trying to establish the leading discourse related to recurring decimals, she takes the class though a rigorous argument using the algebraic routines associated with partial sums (which are each finite) and which culminates in a concept of infinity related to the limit of these partial sums and hence the value of the recurring decimal represented by $0,\bar{9}$.

3. When do these changes occur?

The predominant discourse amongst the learners during the course of the two lessons is related to the process of approaching ever closer and closer to a particular value (interestingly always from below). During the first lesson the teacher does not challenge this view; in fact she supports and uses it in her arguments when establishing a chain of inequalities such as

0,9<0,99<0,999<….1

At the end of the second lesson the teacher attempts to make the final change to extend the discourse to encompass the limit as being the result of the infinite process. This change does not survive for long and the class reverts to their previously held views. But through having been exposed to these fairly mathematically sophisticated views it is possible that these changes will become internalised later when the learners experience limits once again in their mathematical careers. The changes to discourse, i.e. learning, did not occur in any evident way for any of the learners during the course of the lessons other than arguably for those learners who changed their responses to the final question “Is 0,9 recurring =1?” from No to Yes. See Table 8.1.

4. What opportunities for learning are provided by the teacher and to what extent are these utilised by the learners?

The theoretical framework I have chosen uses a change of discourse as evidence of learning having occurred and so an examination of the extent and manner in which discourse changes did or did not occur will result in an assessment of the degree to which learning did or did not occur.

The results of the final worksheet question also act as an indicator of the degree of change in the learners’ thinking. However, it cannot be seen as conclusive proof as it is not supported by further evidence as to the learners’ ability to utilise the changed discourse effectively.

The teacher establishes the discourse of rational number arithmetic at the beginning of the first lesson. She also alerts the class to potential difficulties when applying naïve finite processes to the infinite by explaining Zeno's paradox which clearly is at odds with our experience of the real world.

At the end of the first lesson she creates a situation where 3 distinct incommensurable discourses related to one arithmetic operation (addition) between rational numbers (which are represented as infinitely recurring decimal fractions) are presented. But she does not resolve the situation before the end of the lesson and there are no obvious discourse changes for the class which occur during the first lesson or between the lessons.

At the start of the second lesson the teacher does not resolve the issue related to adding infinitely recurring decimals and thus does not develop a leading discourse for this case. In fact, the class is left without any strategy for adding similar rational numbers except for the approach of converting them to fractional representation and then adding them in this way. Instead the teacher returns to the 3 arguments put forward previously in conjunction with the class to justify that $0,\bar{9}=1$. At this point she reiterates that arithmetic of the finite cannot be transferred as is to the infinite situation and thus the algebraic argument previously used is not rigorous. She also points out that $0,\bar{9}$ looks like "it has something missing" when compared to 1.

At this point she attempts to change the discourse related to infinite fractions by contrasting the situation of a very long but finite representation to an infinite one. She does this by attempting to have the class follow every step of the argument while building a sequence of ever longer recurring representations i.e.

0,9<0,99,0,999<0,9999…..<1

She establishes a change in discourse by arguing that every finite representation of the list above is strictly less 1 and hence in the infinite case which is greater than every one of the finite cases, the value must be 1.

So at the end of the second lesson she does establish a change of discourse which appears to be accepted by the class. However when considering the results of the final worksheet, it is apparent that there is no dramatic variation in the views the learners hold with respect to $0,\bar{9}$. Refer to Table 8.1 reproduced below as Table 8.7.

Table 8.7 Counts of learner responses

Initial answer "YES"	**Initial answer "NO"**	**Changed from :NO" to "YES"**	**Changed from "YES" to "NO"**	**Final answer "NO"**	**Final answer "YES"**
5	9	3	2	7	7

The teacher thus did provide an opportunity for learning at the end of the second lesson by establishing a changed leading discourse but this was not overwhelmingly accepted by the learners.

9 CONCLUSION

This study looked at the extent to which grade 11 learners could appreciate various concepts related to infinity, with especial reference to infinitely recurring decimal fractions. In order to do this they had to grapple with the concept of a limit and change from considering an infinite process of getting ever closer to a particular value (the potential infinite) to the situation of the limit itself as a concrete number (the attained infinity).

Although the class was relatively sophisticated mathematically given their age and grade they appeared unable to internalise and personalise the discourse change which would have indicated that learning had taken place. It has been established that a degree of cognitive conflict was created during the lessons and that learners' faith in the ability to simply extend the discourse of the finite to the infinite was disturbed. However, they struggled to enter any meaningful discourse of the infinite. This is perhaps unsurprising as the historical development of the ideas around infinity discussed in chapter 4 show that the full development of the discourse of the infinite was both a slow and controversial path. Similarly studies by Tirosh (1999), Tall (2001) and Kim (2012) discussed in chapters 4 and 5, although sometimes using different frameworks, have shown that it is difficult for learners to become full and fluent participants in the discourse of the infinite.

It is however, not possible to state categorically that learning did not take place; as given time to reflect, together with further lessons related to the concept of limit over the coming years, the learners may well accept the discourse change that was offered to them during the course of these lessons.

It is also possible that these lessons came too early in their school career for this to be successful. The school curriculum handles limits more fully in the latter part of grade 11 and 12. It would require a follow up study to determine if the class was better equipped for such a lesson later in their school career.

Learners need particular words, endorsed narrative and associated routines involving limits in order to participate fully in the discourses of the infinite. It is perhaps pragmatic to assume that all that can be done in the initial lessons on this topic is to disturb the adherence to the discourse of the finite and to establish to the need for an alternate discourse. Thus learners' full participation in a new discourse as complex as that of the infinite might need to be done in stages with the slow building up of both doubts about the sufficiency of the current finite discourse alongside the introduction of the words and routines necessary to enter the new discourse.

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APPENDIX A

WORKSHEETS

WITS RESEARCH PROJECT – Learners' Conceptions of Infinity:

RESEARCHER: David Merand

Initial Questions

Name:

Grade:

Date:

1. What is meant by a rational number? Provide an example.

2. What is meant by an irrational number? Provide an example.

3. What is meant by a terminating decimal number? Provide an example.

4. What is meant by a repeating decimal number? Provide an example.

5. Convert the following to decimal numbers

   a. $1\frac{1}{2}$

   b. $\frac{1}{8}$

   c. $\frac{99}{100}$

   d. $\frac{1}{3}$

Convert to rational numbers

a. $0,1$

b. $1,25$

c. $0,3$

d. $0,\bar{3}$

e. $0,\overline{54}$

6. Is $0,\bar{9}=1?$ Explain or justify your answer in the space below.

This paper to be handed in.

WITS RESEARCH PROJECT – Learners' Conceptions of Infinity

RESEARCHER: David Merand

Additional Questions

Name:

Grade:

Date:

1. What are the first 7 digits after the decimal comma in the number $0,\overline{57}$?

2. What digit will appear in the 100th place after the decimal comma for the following numbers?

   a. $0,\overline{57}$

   b. $0,\overline{9}$

3. What digit will appear in position 1000, 1001, 1002 and 1003 for the numbers in Q2 above?

   a.

   b.

4. Perform the following calculations and explain your answers.

   a. $0,\overline{43}+0,\overline{34}$

   b. $0,\overline{4}+0,\overline{7}$

   c. $0,\overline{9}\,\text{x}\,10$

   d. $0,\overline{65}\,\text{x}\,4$

   e. $100\div 0,3$

   f. $100\div 0,\overline{3}$

   g. $0,\overline{32}-0,\overline{1}$

h. $1,\bar{7}-0,\bar{8}$

5. Is $0,999$ bigger or smaller than $0,99$? Provide a reason for your answer.

6. Complete the following:

   a. $1-0,1$

   b. $1-0,01$

   c. $1-0,001$

   d. $1-0,0001$

7. Looking at the results of Q6 discuss what you think is the position on the number line of $0,\bar{9}$ in relation to 1 on the number line.

8. Is $0,\bar{9}=1$? Provide a reason or explain your answer in the space below.

WITS RESEARCH PROJECT – Learners' Conceptions of Infinity

RESEARCHER: David Merand

Final Questions

Name:

Grade:

Date:

1. Is $0,\bar{9}=1$? Explain or justify your answer in the space below. This paper to be handed in.

APPENDIX B

TRANSCRIPTS

Classroom Transcript: Learners' Conceptions of Infinity

Lesson 1

Timing/ Speaker	Transcription
Teacher	What David asked me to work with you on is probably one of the most exciting and in some sense mystical areas of Mathematics, which is the notion of infinity. And I hope that some things that happen here will spark peoples' interest and particularly if any of you are intending to continue to university and in Mathematics, there are books and books written on the topic of infinity and discussion about how to deal with the infinite which perhaps created some of the most exciting turning points in mathematics and a lot of us, I think most of you, will by the time you get to matric be doing calculus in your maths class. And the entire calculus, which is essentially the major tool for scientists, engineers and economists, is based in an understanding of infinity. And only could develop mathematically once we had some grip on the notion of infinity. So as we play with the ideas, we are going to start in quite a simple way, we going to start looking at infinity as it quite mystical and I am going to give you a little example, which actually come from the 4th century BC, so it's a fairly long time ago of how people over the centuries have struggled with notions of infinity. And what you will see is, and hopefully what you will come up with as you start to do some of these things, is it will suddenly kind of throw things up that are uncomfortable and don't make

Timing/ Speaker	Transcription
	sense. If you feel like that don't panic because from the 4th century BC right through until fairly recently, even the most eminent mathematicians could not make sense of the mess that comes up when we start with the notion of infinity. So the first little thing I am going to give you is errrr…. I have adapted from this guy called Zeno from the 4th century, who put out these notions which are called these paradoxes and which all related to infinity. So alright, here is my adaption, what you will say to your teacher is that, it is impossible for you to enter the maths class, right? It's absolutely impossible because let's say you start 10 meters away from the door, right? In order to get into the maths class, you have got to walk half of that distance, right? So you walk half that distance, 5 meters, now you still want to get to the maths class, so you…let's start doing a little picture, right?
Minute 3	
	[T draws picture. In the picture: she draws a stick man and a door and uses a line to represent the distance for the learner to get to the door.] In order to get to the door, you have got to walk first of all through the first half of the journey haven't you *[T halves the distance between the stick man and the door by halving the line representing the distance]*. And you're left with half the journey to go. But what happens when you are here? *[T draws other stick man on the point where she halved the line that representing the distance]* You've got to walk the first half of this little piece now, right? *[T then halves the remaining half of the distance and drawing another stick man on that point]*. You have not got to the door so you still got to walk through this last bit but in order to walk through this last bit you've got to first

Timing/ Speaker	Transcription
	walk through the first half of it, right? *[T then halves the remaining half of the distance again and drawing another stick man on that point again].* Can you see how my argument is going to go? Every time I get somewhere, I still have to walk through the first half of it. So no matter where I am, I am still not there yet because I still have to walk through the first half of it. And therefore it is impossible for me ever to get to the door. Yes? Because I have always still got a half a piece to go, and therefore you can tell your maths teacher that is impossible for you to arrive in the classroom because there is always a half of a piece to go, no matter how much. You get why the stuff is now, uncomfortable right? And why Zeno will call this a paradox because we all know, we can walk straight through the door right? But if you think of it this way *[T refers back to the picture on board]*, I have got to keep walking half, half, half, half, and there is always half the distance left so there will always be a little bit left. We have suddenly got a paradox, we know we can get through the door but we have always got half of a distance to go, paradox. And this is what we going to be messing around with your mind, right? Is these notions of infinity and hopefully by the time you finish, some of them will feel slightly less uncomfortable.*[T erases picture]*
Minute 5	
	So the first little bit we need to do in order to get to play the idea we want today is just to check some notions, so would you just have a quick look at the sheet that is in front of you, 1 2 3 and 4, I am going to look at what you are doing and then we will discuss them so we can see is everybody is on

Timing/ Speaker	Transcription
	the same page. *[T walks around room observing learners doing the problems on their worksheets. Learners are working in groups and consulting other members in their group*]
Minute 9	
Teacher	Ok, what is a Rational number? I have seen numbers of correct answers as I was walking around so let's get a volunteer. Ya *[T points at a learner]*
Learner	hmmm...number can be written as $\frac{A}{B}$, when B is not equal to Zero ($B \neq\ 0$) and B is an integers.
Teacher	So when $\frac{a}{b}$, b must not equal to Zero ($B \neq\ 0$) and the other thing we have got to have is that, both (A and B) of them are integers.*[T writes on board Rational number* $\frac{a}{b}$ *, b (b≠ 0) a,b integers]*. So she has given us a very formal way of saying it, right? She has dotted all the i's and crossed all the t's, right? A lot people that I saw said it is something that can be written as a fraction and that how kind of how you got the core ideas there, that can be written as a fraction. But can you see that this is a little more mathematically beautiful, right? *[T points to writing on board]* Because if we say it can be written as a fraction, you know? Then does that, $\frac{\Pi}{3}$ count as a fraction? *[T writes* $\frac{\Pi}{3}$ *]* Is it fraction?
Learner	Yes
Teacher	Yes, and is it a rational number?
Learner	No
Teacher	No, you see why "fraction" is a little bit vague. So this is very beautiful right? So this is a lovely definition of a Rational number and a lot of you that I saw have.got the key idea

Timing/ Speaker	Transcription
	.ideas, things can be written like Half *[T writes* $(\frac{1}{2})$*] 7 over 53 [T writes* $\frac{7}{53}$*]* and anything that can be written as a "normal" looking fraction, right? Let just get to, hmmm... get to irrational numbers. *[T writes "irrational number " on board]*What are Irrational numbers? Ya (pointed at a learner).
Learner	Negative square root?
Teacher	Ok now there you are bringing out a totally...hmmm...other one here. You are taking us out of what I wanted to hear. You are taking us to imaginary numbers.
Learner	Oh ok
Teacher	Because what you are getting us to is exactly correct and if you try to take a square root of minus one *[T writes imaginary numbers* $\sqrt{-1}$*]* in the world that we live in, that is just not possible right? We have to move into a whole different number system to do that. Ok but irrational numbers are numbers that exist in the world of numbers as we know them. Give me some examples. Don't be shy; there are a lot of correct answers. Ya *[T points at a learner]*.
Learner	Pi *[T writes π]*
Teacher	Pi (π) is a beautiful irrational numbers, what others? Ya *[T points at a learner]* .
Learner	hmmm...any square root of one.
Teacher	Yes, if you got...
Leaner	any square root of a non square.
Teacher	Yes, any square root of a non square, exactly right. So if you got square root of a 2 *[T writes* $\sqrt{2}$], square root of 3 *[T writes* $\sqrt{3}$], obviously square root of 4 is 2 so. And ammm... what is going to happen to their decimal representations? If

Timing/ Speaker	Transcription
	we turn them into decimals, what do we get? Ya *[T points at a learner]* .
Learner	The numbers that goes on forever.
Teacher	Exactly, what you get is a whole string of decimals but you will never get any pattern occurring in those decimals that repeat right? So it never ever stops itself. And you can hear in this, a lot of the story of the development of maths. I mean rational number and irrational numbers; you can hear all the prejudice against irrational numbers right? *[T points to written words on board]* And that is what happened right? Because way back in the time when Pythagoras where they started to come across this ideas of irrational numbers for the first time, they looked at a triangle and they said one, one those are nice numbers and easy to understand and when they calculated the hypotenuse, it came out as root 2 *[T draw triangle with sides labeled 1 and hypotenuse labeled $\sqrt{2}$]*. They played around with that and they couldn't find a number that they knew that root 2 was equal to. And it really freaked them out, right. And so this notion, it carries through to today behind the term irrational, there is a fairly strong prejudice because they are numbers we don't like, because they don't behave very nicely for us and they don't start repeating.
Minute 13	
Teacher	Terminating decimal, what does Terminating mean as a word? *[T points to word terminating written on board].*
Learner	Comes to the ends
Teacher	Ends right? Stops right? Like stopping decimal, give me some example.

Timing/ Speaker	Transcription
Learner	3 8
Teacher	3X, right? *[T writes 3x on board]*
Learner	3 8 three eighths *[T writes $\frac{3}{8}$].*
Teacher	Oh, 3 8 ($\frac{3}{8}$) yes, 3 8 ($\frac{3}{8}$) is going to give me a terminating decimal. What is the terminating decimal going to be?
Learner	0,375
Teacher	*[T writes $\frac{3}{8}$ = 0,375]* right, Is a decimal that stops after a finite number of steps right? Plenty like a half is 0, 5.*[T writes $\frac{1}{2}$ =0,5]* Err... Repeating or Recurring decimal. Ya (pointed at a learner).
Learner	hmmm... is decimal number that carries on forever with same pattern.
Teacher	With the same pattern that exactly. So you get some very boring *[T writes 0.333333333 on the board]* which is just the 3 going on forever and ever. But you get some, that start off looking as if they are not doing anything nice *[T writes 0,2754293293293...]* and then they start to repeat. Right? So anywhere where you've got some repeating happening right? Just to get our notation the same because people, I use dot or bars right, to represent it. We have, in this worksheet used bars just cause it prints out better, the dots sometimes disappear when you photo copy. So that's what we mean, right? You might have seen it in other textbooks as that *[T writes 0,33333… = $0,\overline{3}$]* right? Is just the same thing, there is nothing fancy and here what we would do, all things that first appear that aren't repeating, we just put them in there and then if we put the bar over all of those *[T writes 0,2754$\overline{293}$]* it means from this point on, you just

Timing/ Speaker	Transcription
	repeat 293 293 293. Just one more thing, I saw some people had different opinions on this *[T points at the topic Rational numbers on board]*. Rational numbers, are these things *[T points at Terminating decimals]* and these thing *[T points at Repeating decimal]* rational numbers? Let's slow the question down. Are terminating decimal going to be a rational numbers?
Learner	Yes
Teacher	Any terminating decimal?
Learner	Yes
Teacher	Completely convinced?
Learner	Yes
Teacher	Cause you can always turn terminating decimal into a fraction form. So these *[T points to Terminating decimal]* are definitely rational. Are terminating decimals the only kind of rational numbers?
Learner	No
Teacher	So my question is now, are repeating or recurring decimal are they rational?
Learner	Yes
Teacher	Ya, you should have played at some point with being able to turn a recurring decimal into a fraction form. And so this one *[T points to 0.333333333 on the board]* is an easy one, which i know is going to be 1 over 3 [T writes on board $\frac{1}{3}$) so in fact, rational numbers contain both the terminating decimals and repeating and recurring decimals. *[T draws line from rational numbers to terminating numbers and repeating decimal]* I just saw one who I think was talking about terminating decimals as if they were the only rational

Timing/ Speaker	Transcription
	numbers, and it's not right. There are plenty more, right. *[T points at repeating and recurring decimal]*. Are there any questions on this? Or is everyone clear? We just need everybody to be clear on various types of numbers. Everyone ok? Ok! So let's continue to 5 and 6, and I will have a look of what you doing with that. And as soon as you finish on that, I want you to start discussing question 7, and for question 7 I want everybody individually to write a solution. Alright, I want you to discuss it together but I also want individual, you know I want to collect everybody's individuals' solution at the end. Ummm...cause that's the most interesting question to ask, so start with 5 and 6 so that we can see everybody is on track there and then as you can get through quickly as you can and start on 7.
Minute 23	
Teacher	Let's go to question 7 because I hear fascinating discussion, so we are going to get some different ideas from the groups. But let's just quickly get through, cause we are going to want to have these methods is, that one *[T writes* $0,\overline{3}$ *written on the board]* turning that one into a rational number form and that one *[T writes* $0,\overline{54}$ *on the board]*. Who would like to do the 0,3 recurring? 'Cos most people got the answer, some using memory but there were quite a few who couldn't recall the method for doing this, so can we have a volunteer? Or should I volunteer a volunteer? This table had someone who was doing it. Here we go, a volunteer.*[T hands pen to learner]*
	[The learner approaches the question on $0,\overline{3}$ *and writes on the board]:*

Timing/ Speaker	Transcription
	[Learner writes Let X=0,$\dot{3}$ 10X = 3, $\dot{3}$ 10X-X = 3, $\dot{3}$ – 0, $\dot{3}$ $\frac{9X}{9} = \frac{3}{9}$ X = $\frac{1}{3}$]
	Class applauds
Teacher Time	Right, anyone see what she did in the method? And how did this one differ then? *[T points to* $0,\overline{54}$ *on the board]* differ then? Ya
Learner	Instead of multiplying X by 10 multiply it by 100.
Teacher	Perfect, right? So here you will have X *[T writes on the board,* $0,\overline{54}$ *= X]* and this will became 100X and this side is equal to $54,\overline{54}$ *[T writes on the board, 100X =* $54,\overline{54}$ *]* and then we would be left with 99X and this side is 54 *[T writes on the board, 99X = 54]* and the X is 54 over 99 *[T writes X* = $\frac{54}{99} = \frac{6}{11}$ *]* and of course some people cancel down into $\frac{6}{11}$. Ok so just wanted to check that we all had that method somewhere in our heads, that was probably taught from earlier, I am hoping. Ok let us now, but please don't be shy about this because at every table there was interesting discussion going on around the "Zero point nine equal to one" *[T writes* $0,\overline{9} = 1$.*]* Right. Let just have a quick show of hands, who votes… I mean who said zero point nine recurring is equal to one? (About 30% of the learners raise up their hand), ok and who said it wasn't? (About 70% of the

Timing/ Speaker	Transcription
	learners raise their hand). Ok you can see you don't have to be shy because you have got friends either way, you won't be out there on your own if you said it is not equal to one or it is equal to one. And let me tell you, this is one of the huge debates in Mathematics for a really really long time it was part of the crucial debates about things so you are in a company of many mathematician over time and at least a section of the class agrees with you so please let us hear what your thinking was, why you said what you said? Alright? I want to get one person from each group, so we will start with the volunteers and I will start volunteering people so let's have some people. And you are welcome to come up to the board and use pens if you need to or say what you need to. So lets' starts, is there any volunteers or should I choose? (When the class teacher walks in, the teacher says, "Well, you have come to the interesting parts. We are about to discuss whether zero point nine recurring is one and they have all discussed it and each group has come up with some idea, so now they are going to tell us. Ok now *[T then points at a learner]*.
Minute 27	
Learner	Well I say zero comma nine is equal to one because if $\frac{1}{3}$ is equal to $0,\overline{3}$ and $0,\overline{3}$ times 3 is $0,\overline{9}$ that would make $0,\overline{9}$ one because 3 times $\frac{1}{3}$ is 1. Gasps from class members.
Teacher	The teacher wrote on the board: $3\ X\ \frac{1}{3} = 0,\overline{3}\ X\ 3$ $1 = 0,\overline{9}$
	Alright, here is argument number one. As produced by….oh by the way, this table didn't necessarily agree with him by

Timing/ Speaker	Transcription
	the end did they?*[T draws box around equations and labels box 1]*
Learner	Shakes his head to confirm.
Teacher	So there is his little argument, right? May we have some opposition? Yes *[T points at a learner]*.
Learner	Well there's a counterargument, which is from like as opposed to doing it as if we think of it out of maths as a mind problem. Like the way you have done it by numbers of steps to the classroom. There is always…no matter how many halves you take, there is always a tiny little bit away from it so maybe it is infinitesimally small but there is a difference between one…and ya one like...
Teacher	So there will always be a little bit missing. Right, ok. So there is one of our arguments for "no". Any other people? Ok what is your thinking at the back there?
Learner	Umm… ok so we used a method let X equal…
Teacher	Come do it on the board and do it. Come lets' try it.
Minute 29	
Leaner	let X equal to zero comma nine recurring *[L writes on the board, Let X =* $0,\bar{9}$ *]* and then 10X minus X equal to zero comma nine recurring minus zero comma nine recurring *[L writes on the board, 10X-X=*$0,\bar{9}$-$0,\bar{9}$*]*, so it left with nine X equal to nine *[L writes on the board, 9X = 9]* and if we divide both side by 9 *[L writes on the board,* $\frac{9X}{9}=\frac{9}{9}$*]* then X will be equal to one *[L writes on the board, X=1]*.
Teacher	Ok, right. So you people split your own heads. So you are saying "Here is our argument that it is one but we don't believe ourselves".

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Learner	Because if you have to add a zero comma zero zero zero …. 1 then it will be equal to one *[L writes on the board on the board, 1 =* $0,\overline{9}$ *+ 0,00000…1]*. So it doesn't look the same.
Teacher	Ok so there we have the split personality group, where they both believe it and don't believe it. So they had two arguments. And this is the second argument as to why they believe it *[T draws box around equations and labels it 2]* And this argument *[T points at 1 =* $0,\overline{9}$ *+ 0,00000…1]* is similar to your argument *[T points at the previous learner that spoke]*. Which is saying…I don't know, maybe I should fill it in green for the argument, ok this one is similar to the first argument and say there's a bit missing. Ya
Learner	Can I ask hmmm…..how do you write it as fraction? Iif it's recurring then it should be written as fraction so how do you write it in fraction?
Teacher	I am not answering any questions today. I am wanting to hear the discussion today right. This is what we need to struggle with and try and figure out alright? So what you asking is an excellent question for us. You have come up with a very deep mathematical idea right. You are saying if its recurring then it should be something that we can write as a fraction isn't it. So what? Ya *[T points at a learner]*.
Minute 31	
Learner	Isn't one over one is an fraction? (another learner said: yes one over one is an fraction).
Teacher	So she is saying X = 1, that 1 is an fraction *[T writes on board,* $\frac{1}{1}$*]*. Ya so let's just listen to what she is saying, you

Timing/ Speaker	Transcription
	have asked a very important question, "…it should be able to be written as a fraction" so she said, look we have done it. We have taken zero comma nine nine nine forever and we have gone through this process and got to an answer of one *[T writes the board, 0,99999 = 1]* from the process above and that is an fraction *[T writes the board, $\frac{1}{1}$]* and its unusual fraction but it's a fraction nonetheless right? So we are happy. And then she said it also feels uncomfortable, cause if we continue this *[T points to $\frac{1}{1}$]* and turn this fraction into a decimal, it becomes 1,000000 ya? And so suddenly it seems quite yucky, right? Arguments over here? What other argument? I am leaving things hanging alright, I promise you by Wednesday we will start to pull things together but this is this playing with your head, starting to play with infinity. Ya *[T points at a learner]*.
Learner	Can I…go back. Would you argue the same thing that we argue that One comma zero zero zero zero zero zero zero one (meaning 0,000000………..1) with a one infinitely at the end is equal to 1? I mean you have to argue that that is one because it's the same thing.
Teacher	*[T writes on the board, is 1,000… (Infinitely many 0's)...1 = 1?]* So that's his question to you, right? Is one comma infinitely many zero's one also equal to one? (The class starts laughing). You see why there are books written on this subject? Yes, let's hear from this back two tables *[T points at the back]* just wanted some of your thinking along. Ya *[T points at a learner]*.
Minute 33	

Timing/ Speaker	Transcription
Learner	I thought it was equal (the learner mean $0,\overline{9}$ equal to one) but now I don't think so anymore. (the whole class starts laughing again) because I thought you guys… cause if you look at a third? And you put three over nine. And I thought most things over nine normally recur so I just put that nine and assume is one. But now after listening to them, I just … (shaking his head). Another learner added in, "not so many". And then the initial learner said, "Ya".
Teacher	Yes, so you were looking at the pattern, you know?
Learner	Ya
Teacher	Three over nine, which is third zero point three da…da…da…da. So you thinking over nine, and nine over nice should be give you the answer, ya. So the pattern is suggesting that that is what it should happen. But now you starting to doubt. Ya *[T points at a learner who was the first to give the argument that* $0.\overline{9}$*.= 1]*
Learner	Those who think that zero comma nine does not equal to one, which I think is kind of right now. Umm…It wouldn't leave the zero to one, it just some that is impossible. Like divide one by three, possibly. So it actually…two third.
Minute 34	
Teacher	Just…just explain a little bit more because I don't know if everybody else is following so what are you saying?
Learner	if…if…if it's going to be zero comma three three three three recurring to infinity.
Teacher	Ya
Leaner	if you have to multiple by three, it will be one. So…**(?)**…
Learner	That's true! Yea, the fact that you get zero comma three recurring together but you can never get the perfect term

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	because they is always another decimal three to make it that much closer. So we could never actually reach a nice round number at the end.
Learner	Yes!
Teacher	Ya (pointed at a learner)
Learner	But also is easier with three because you can actually write three as an fraction…I mean third as an fraction. Whereas like the same as zero comma nine recurring can't be a fraction so we get one over one. So I think it is just easier with the 3.
Teacher	Ya so three is certainly feels more comfortable to say it's a third, ya. Ok, but it still make…I mean you point is interesting, so let's not a lose you point there right? You say… I mean it basically telling us this is the key right? *[T points at 0,$\overline{3}$ on the board]*. Saying if you look at zero point nine, the third of it is zero point three recurring. It feels right and it should then be one when multiplying the third by 3. [T pints to $3\ X\ \frac{1}{3} = 0,\overline{3}\ X\ 3$ $1 = 0,\overline{9}$] Our very back table there.
Learner	if you take…ok I tell you now that if there is a ninety nine point nine nine nine nine nine percent of chance that will not be a hundred percent. But there is still that uncertainty that can be, even is that there is that zero point one percent or whatever, there is that possibility. So you can't… like… totally ignore it and say that… err… zero comma nine is just equal to one because that there is still that difference.
Minute	

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36	
Teacher	Ok I am going to…. I am going to leave this discussion here and we are going to bring it up again on Wednesday, alright. So I am going to leave it to churn, we are going to play a little bit more with some of these infinitely recurring decimal to see what's happening. But I do just have to say you are Grade 10 right?
Learner	Yes
Teacher	I had this discussion with third year university students and let me tell you, you people have come up with more sophisticated understanding and reasoning then a third year class that I was doing so I am unbelievably impressed with you. So I am hoping, let's do the next couple of these things and tackle some more of these things, paying with these recurring decimal and then I am very excited to see what will come out and our discussion and then let me follow up on Wednesday.
Minute 43	
Teacher	Ok let's go to Question B, not from those who people who is doing the nice thing of working with the fractions. But for people who is playing with the infinite decimal and leaving them as such, I have seen in three answers right? So here have look and you guys can carry on discussing in your group. So one group said, look I am just going to write the sum like this. I go four plus seven is one, carry the one so and I get one point one and of course the bar stays right? So its recurring.
	She wrote on the board and labelled it as number one, $0{,}\bar{4}$

Timing/ Speaker	Transcription
	0,$\overline{7}$ 1,$\overline{1}$
	Then another groups and said ok, alright let's write it out and you go four plus seven is one and carrying on the one and then four plus seven is eleven plus the one from the previous is twelve and that is going to keep on going right? So I will get…so what I have got is sort of a funny thing with a whole lot of two's, infinitely many two's and a one at the end right? So this (1,222…21) is what they mean over here infinity many twos right?
	She wrote on the board and label it as number two, *0,444…44* *0,777…77* *0,222…21*
	And the back table and they can probably explain it better than I can, said no no no no the answer, errrrr not zero here, is one (She corrected her previous answer by changing, 0,222…21 to 1,222…22). The back table said it is not going to be like that (pointed at 1,222…22) it is going to be one comma two recurring *(Written on the board and label as number three, 1,$\overline{2}$)* because this (as mean the infinity of twos) is going to go on so it is keep going to be one point two recurring because this is just going to go on, right? So you will keep on getting the twos' so in fact it is going to be one point two recurring. Maybe the back table can explain better than I am. Is that ok? So now we in that position where I have seen three plausible answers to question B. So I am throwing that back at you, as part of the lesson

Timing/ Speaker	Transcription
	which is a lesson of confusion. I am throwing those three answers back at you, for you to discuss as well.
Minute 45	
Teacher	You want to take… maybe take a few points and then leave some for you to take it home. So let's take a few points. Yes (pointed at a learner).
Learner	Well ours is wrong so is it.
Teacher	Ok, which is your one?
Leaner	The first one
Teacher	Ok, so this one is your one *[T points at number one]* and you are saying is wrong. Why?
Learner	It's just is, our method is bad, ya.
Teacher	But I mean it seems ok, when we still do… (Written on the board $0,\overline{3} + 0,\overline{3} = 0,\overline{6}$) that will be ok.
Learner	Because that number when I added together it did… it give another value.
Teacher	Ok it seems all is ok in some cases, but if there is a carry then it is not ok? Ok so they are saying their answer is incorrect. Anybody else? Oh…you happy to throw them out, ok, alright. (The whole class laughs a bit). Ok what about these two? *[T points at number two and three]*. Ya *[T points at a learner]*.
Learner	We…we turned it into an fraction so it have the same…hmmm… denominator so it will be nine after we add it.
Teacher	So like this four ninths, seven ninths equal eleven ninths. *[T writes on the board, $\frac{4}{9} + \frac{7}{9} = \frac{11}{9}$]* and that is a beautiful solution, alright? That is lovely but for… but you see right, I

Timing/ Speaker	Transcription
	am wanting you…I am wanting to mess with your mind, right? And unfortunately fractions are too easy, *[T crosses out fractions written above]* I can't mess with your minds with fractions so please could you mess with your minds and do it *[T points at examples labeled number two and three from previously]* so don't forget your fractions because they are brilliant right? And it's a very good solution you gave, but now mess with your mind with the infinite decimals and tell me what's going on (pointed at number two and three). (Pointed at a learner).
Learner	Well, I just say it's for number three because it goes on for infinity and infinity doesn't have an end.
Teacher	Ok (pointed at another learner)
Learner	Well, basically it's the same thing as the zero point nine recurring question… err… get because err…is the same thing with no difference, I mean it's just a difference of a zero and a one. So it's really I don't think it can be resolved. Hmmm… two ordinary such as… **(?)**…
Teacher	Ok, these are interesting points and I am deliberately not judging at this point because what we are doing is bringing out some of the interesting debates which are important for all of us to think about it.
Learner	Well, it depends how you imagine infinity because if you imagine infinity as almost circular, like the drawing of infinity (he show sign on his hand ∞) then it will be number three but if you imagine infinity as linear so carrying on forever and ever and never kind of connecting up like that then it is number two. So it is how you are imagining infinity.
Teacher	Lovely, ok, back of the class, yea *[T points at a learner]*
Learner	Well, err… number two, how do you know that infinity ends

Timing/ Speaker	Transcription
	with one?
Teacher	How do you know?
Learner	That infinity ends with one?
Teacher	Ok, so how do you know that infinity ends with one? Ya, there is a sort of…ya ok let's go.
Minute 48	
Learner	I think that number three has to be sort of correct because umm… you it can't end and for it to end at a one, I means it has ended and there is nothing afterwards. So I think it has to be number three because that's the only way that…you know that… infinite doesn't actually end and it's continuously moving on.
Teacher	Anyone else have a point? You guys are bringing lovely points out, I mean. Hey, let me let you loose back on these things. What we are doing is throwing up things and I hope will churn through your head for the next few days, right? And you will start, you know kind of messing these around in your mind for a bit and we will try to draw time for this on Wednesday.
Minute 59	
Teacher	Ok, won't you just make sure that you guys given us your paper and with your name on them. Thanks very much I find that the discussion was fascinating and I will see you guys on Wednesday.

Classroom Transcript: Learners' Conception of Infinity

Lesson 2

00:00 **L (T):** Just a reminder, my name is L in case you want to call me anything and we're going to carry on with where we left off, although possibly a bit more of me chatting away this time. But before I start to chat I want to just find out if any of you had any further thoughts or any comments you wanted to make on what we did last week. Anything to lead us with? Any ideas?

[male learner 1 (ML1) raises his hand]

T: Ja?

00:28 **ML1:** Infinity is crazy.

T: Infinity is crazy. *[T laughs]* Ja, I think there'd be a lot of people who would agree with you on that one – infinity is crazy. Let me remind you perhaps then a little bit of what we did and some of the arguments. Our core question was: Is … equal to 1? Right? *[T writes the question on the board while she talks:* 0,999… = 1?*]* Is 0.99 equal to 1? And you guys produced some incredibly interesting arguments. And that was amazing because as I told you last time, the university students that I teach hadn't been able to produce those arguments. I showed them the arguments, remember, they hadn't been able to produce them. So you guys really operated at a very high level on this. So let me go through a few of the arguments and then we'll carry on from there.

So one of the arguments is our very standard little story that you're used to *[T writes: x = 0,999]* ***[this becomes Argument 1]*** in terms of dealing with infinite recurring decimals which is if we want to convert that into a rational a over b style we say, 'Ok if we've got x as being 0.9 recurring what will $10x$ be? *[T writes:* $10x$ = 9,999 *]* Right, we're just multiplying by 10 so it becomes 9.9 recurring. We say $10x$ minus x and we get $9x$. *[T writes:* $9x$*]* We say this minus this *[T points to* 0,999 *and then* 9,999 *on the board]* and we get all these disappear with each other and we get 9 *[after* 9*x T writes* = 9 *]* And so we conclude that x is 1. *[T writes:* $x = 1$*]*

And so what have we done? We've said x is 0.9 recurring, we've done our little process and we get x is 1 *[T writes:* 0,999 = 1 *]* so this little argument would suggest that 0.9 recurring is equal to 1.

02:22 **T:** We then had a second argument that came forth from you *[T addresses learner 1 (L1)* I don't know your name. Sorry, I don't know anybody's name. Your name?

L2: Mxxxx

T: Mxxxx gave us this argument which is a lovely argument. ***[This becomes Argument 2]*** *[T writes:* 0,333… = ⅓ *]* We know that 0.3 recurring is a third. We can multiply both sides of this equation by 3, so if we multiply this side by 3 *[T writes:* 0,999… = 1 *]* we get 0.9 recurring. And we multiply that side by 3 we get… And so here's another little piece of maths that seems to suggest 0.9 recurring is 1.

03:06 **T:** And then if we just have a little look at… if we start playing with patterns *[T writes:* 0,111… = $\frac{1}{9}$ *]* ***[This becomes Argument 3]***

you know that one ninth when you convert it is 0.1 recurring and turn 0.1 recurring into a rational form you get one ninth. Ok. (Is there another pen here somewhere? Oh thank you, mine seems to have run out. Thanks) *[T gets another pen]* 0.2 recurring is what? *[T writes:* 0.222… = *]*

A learner: Two ninths

T: Two ninths, right. *[after* 0.222… = *T writes* $\frac{2}{9}$ *]*

0.3 recurring is three ninths or one third *[T writes:* 0,333… = $\frac{3}{9}$ *]*

If we carry on with that pattern we get up to 0.7 recurring is what?

[T writes: 0,777… = *]*

A learner: Seven ninths

T: Seven ninths *[after* 0,777… = *T writes:* $\frac{7}{9}$ *]*

0.8 recurring is what? *[T writes:* 0,888… = *]*

A learner: Eight ninths

T: *[after* 0,888… = *T writes* $\frac{8}{9}$ *]*
Eight ninths *[T writes:* 0,999… = *]*

And 0.9 recurring then has to be what? If we follow the pattern it seems that we really should call it 9 over 9 which is 1.

[after 0,999… = *T writes:* $\frac{9}{9}$ = 1 *]*

04:21 **T:** So here are 3 little playing with some mathematics all of which end in the story 0.99 recurring is equal to 1. So my question to you is let's label them 1, 2 and 3 *[T labels the 3 different methods]* so we can talk about them easily. Argument 1, Argument 2, and Argument 3. Are any of those arguments convincing to you and do you have problems with any of those arguments? Right. So let's just… Let me give you 2 seconds. Lean to the person next to you or behind you and just talk to the person next to or behind you. Any of these arguments or all of them do they convince you 0.99 recurring is 1 or is there something in these arguments that worries or ? something? Talk to someone for a couple of seconds and then we will… *[learners discuss this]*

06:20 **T:** Ok, let's share more broadly and let's hear what you discussed, what you thought. Anyone going to start us off? Any of these arguments convincing or is there something problematic in them? *[T points to a learner]* Yes?

Female learner 2 (FL2): In argument 3 it says 0.99(?) recurring equals 1 so it's kind of like rounding off, but does 0,888 equal 0,9 because 0,9 is in a fraction like represented as 9 over 10. And 9 over 10 does not equal 8 over 9.

T: Ok, so you're feeling like this is… This feels to you like they're rounding and then you'd want this one to also round up to the nearest which you say would be 0.9. Right, so that's the way you're understanding this argument. Ok, fine. Anybody else have comments?

[ML1 raises his hand]

T: Ja?

07:07 **ML1:** Just like, sort of like an argument we did last week was if… Ok, so 0,9 plus 0,1 that's 1. 0,99 plus 0,01 is 1. So is 0,9 to infinity plus 0,000 with a 1 where you see it continuously into infinity, would that equal 1 or would that equal 1,00001 proceeding to infinity?

07:36 **T:** Do you get what he's saying?

Class: Yes

T: Lovely little argument there. We've got 999 that goes on forever and then he's saying if we add on this sort of thing *[T writes something on the board which is not visible on the video but I think it is*

$$\begin{array}{r} 0{,}9999\ldots \\ +\ \underline{0{,}0001} \\ 1{,}0000 \end{array}$$

]

and these little dots must represent going on forever, do we get 1 or do we get that? *[next to* 1,0000 *T writes something which is not visible but could be* or ? *]* Right? Yes. Other comments? *[T points ML3]* Ja?

08:06 **ML3:** Like you look at this and it all makes sense, but at the end of the day you're looking at 2 different numbers that shouldn't be equal to each other.

T: Right. So you're saying each step of these arguments makes sense to you, but the overall conclusion still doesn't feel happy.

ML3: Yes

FL4: But Miss, ML1's, ML1's argument doesn't really make sense…

T: Ok

FL4: … because it ends. 00001, then there's still 9s beyond that. So it…

ML1: There's 9s here(?) too(?). In the same way that the 9s continue into infinity, it's always there and it's always 9s but there's always zeroes with a hypothetical 1 at the end.

FL4: But then there's still an end.

ML1: Not necessarily, it could be a continuum like moving.

T: *[T addresses FL4]* He's got a vision that you say is different to your vision, right, so I can hear what you're saying. Absolutely. When you put that 1 there you're making an end to the number. *[T points to ML1]* His vision says it doesn't really end because what he sees, he's got this 1 there but an ever expanding middle bit that keeps pushing the 1 further and further out. Ok, but ja, nice, because this is a nice discussion, right. What did you say at the beginning? *[T looks at ML1]* Infinity is crazy, right?

ML1: Yes

09:27 **T:** And it is, right, because you're giving us one version, right, which makes a lot of sense to me – once you put the 1 on the end you're stopping things. You're giving me another version which is this bizarre idea of the infinitely expanding middle in a number which ja, goes with your 'crazy' notion.

[ML1 raises his hands in triumph]

T: Other comments?

09:47 *[FL5 raises her hand]*

T: Ja?

FL5: We agree with Mxxx and ML1 because ML1 has the 1, I mean the zero comma… zero zero 1 at the end. And if you add that onto 0,9 recurring maybe you would get 1. But they're not equal because there is a difference between the two of them. If there's a difference between the 2 numbers then obviously they can't be equal to each other.

T: And how do you see the difference between the 2 numbers?

FL5: Because they're not the same?

[class laughs]

T: Ok, so…

Female learner: 1 doesn't

10:19 **T:** Alright, we're getting some lovely ideas here. Let me take a few more. I'm just trying to remember all the things I want to say to sum up these ideas. Any more ideas? Comments? Because you've come up with a few things which are very nice here, right. The first is that you're saying one of the things that doesn't feel nice is there two things look completely different. *[T points to the board]* And so it just doesn't feel like they should be the same number. So now I'm going to just… Let's work on that idea quickly. *[T cleans the board]*

11:03 **T:** Actually let me say something else before we work on that idea. The other one, right, ML1's little notion here *[T points to the board]* should ring bells with some of the stuff we did last time. The minute we started playing with these infinite decimals it was fine for a while but then when we… so it was fine doing things like multiplying by 10 and multiplying this thing here by 3 *[T points to the example on the board]* But remember when we tried to multiply this thing here by 4? If we tried to go 0.33 recurring multiplied by 4 *[T writes:* 0,333… x 4*]* if we tried to do it in the decimal form we got ourselves into a little bit of trouble, right, because we couldn't quite figure where the right place was to start. Is it 4 times 3 is 12 *[T writes on the board, but it's not visible]* carry 1, then 4 times 3 is 12 plus 1 is 13 carry 1, right? Do we end up with 1.333 *[T writes: 1,33.3…]* with a 2 somewhere at the end or is it just 1.333 recurring, right?

12:06 **T:** And we had similar, we could say similar stuff if we started to try and add something like

[T writes 0,777…
+ 0,333… *]*

those two things, ja? We would have a similar problem, right? Because when do you, where do you start the addition from? And you're basically telling us there's the same sort of story here, right? *[T points to the board]* That we're not quite sure what the other little pieces here and where do you actually start from. So you guys are pointing out a very important point to us here which is that we actually can't be a 100% sure how addition and multiplication work when we've got infinite decimals.

12:55 **T:** What have we done? We've just assumed that once we've got infinite decimals that the multiplication works the same way it's always worked, right? So this argument here *[points to Argument 2 on the board]* is based on the idea that we can just multiply each of the little 3s and get to the 9, right? We're just making an assumption that when we work with an infinite amount of things it works the same way as we work with the finite, right? But we saw if we continued a little bit that we end up with some nasty cases where we can't necessarily be so sure of that fact. And that makes all of these arguments just a little bit unconvincing, to me anyway, because we don't know that you can actually do the multiplication in there. But let's deal with that as we go on.

13:52 **T:** What we're starting to get a pattern of is that things when we get to infinity start behaving in ways that we maybe are not as sure of as we would be if we were just dealing with finite numbers.

14:08 **T:** The question of the thing looking completely different, 0.99 recurring looks completely different to 1. How can they be the same number? This feels like you've got something missing. *[T points to* 0,999… *on the board]* whereas this feels like a whole 1. *[T points to the 1 written on the board]* So my question is, if you're feeling that way why are you so convinced of that fact? *[T writes:* 0,333… = ⅓ *]* *[T cleans the board]* And does 0.333 recurring really equal a third or is it maybe just a little bit less than a third? I mean they're also two totally different looking numbers, so why are you convinced? Nobody's queried that one yet. *[T addresses FL4]* Yes?

14:56 **FL4:** Because 1 divided by 3 is 0,333 recurring, but 1 divided by 1 is 1.

T: Ok, but now why do you know that 1 divided by 3 is 0,333 recurring?

FL4: Because you do the long division and… *[laughs]*

T: Ok, I'm going to ask you to come and show us.

[ML1 raises his hand]

T: Yes?

[class laughs]

15:18 **ML1:** Well it's the closest thing we can write in our ten numeral system that can get to a, to a third. But it's not, it's not that, that's why it continues into infinity. It's infinitely closer than ? though. So say we had… If we had a system instead of like tens (so our whole number system works in 10s) if we worked in 9s it would be much simpler because we could have a 9 ? but then 2 ? a path ? that continue like this to infinity as well.

T: Mmm

ML1: So it's… It's the way… it's our number system, means that that's the only way we can show it, and even that's not quite.

16:00 **T:** Ok. So you… *[T addresses the class teacher?]* I mean where did you find these students? *[laughs]* You guys really have some brilliant ideas here because that's exactly… You've got exactly the right idea that it is reliant on the fact that we're working in a decimal number system, a base 10 number system. That we end up 3 doesn't(?) divide nicely into 1, there's no neat representation so you get your… *[speaking to FL4]* Do you want to come and do your long division for us. No? *[laughs]*

16:28 **T:** So your argument is… *[T does a long division sum on the board]* The reason why you know for sure is that when you go 3 into 1 you get 0, you carry the 1. 3 into 10 goes 3 times, which is 9, you get a remainder of 1. You go 3 into 10 again, it's a 9, you get a remainder of 1. You keep passing over the 1s so you're going to keep on getting 3s and we just put the dots to say we're going to keep on getting it. Alright?

The sum:

```
  0,333…
3 | 1,000
     9
     10
```

Now we have a doubter at the back who says that 0.333 recurring is not actually a third, it's kind of a sort of close to a third, it's the best representation we can give of a third. Believe him or not?

[FL6 raises her hand]

T: Ja?

17:15 **FL6:** I think yes, because the whole reason you carry on going is because you know you haven't got there yet because you haven't got something that doesn't leave a remainder anymore. Like 24 divided by 6 is 4.

T: Yes

FL6: Um, you know, but if it was 25 divided by 6 you would have to carry on going. So it's not, it's not something that stops and there is no longer a remainder. So ja, you never get quite close enough to it. You never get the exact value but you get the closest.

ML7: Ja, I was just thinking that means that 1 or 2 cannot exist *[some laughter]* because ? would mean that the 1's position of something, maybe that would equal that and keep on adding something to the one number of the 3 in some way to actually make it equal to… Seeing as 1 doesn't stop, just adding on 0s, so one of the 3 doesn't exist.

18:12 **T:** Ok, so now we really are getting to the realm of, you know, now the number 3 doesn't exist at all anymore. *[laughs]* Ok, so we're destroying systematically every little piece of mathematics that we know. Our comments on is 0.333 recurring equal to a third?

[FL2 raises her hand]

T: Ja?

FL2: Well in English a third means 1 divided by 3.

T: Yes

FL2: Which is the exact same thing as saying the long division one divided by 3. So 1 divi… I mean a third is not exactly a whole number, so you can't assign… Ja... *[laughs]*

[class laughs]

18:51 **T:** I know what you're trying to say.

FL2: So then like one of the… Like the third, you call it a third, it's not a whole number so it's not really 'finished' or 'ending'. So it should equal 0,333 because 0,333 is not ending. It's not ending like a third, ok. *[giggles]*

T: The not ending is precisely our problem here.

[ML1 has raised his hand]

T: Ja?

19:16 **ML1:** I think that it's not that you can never get a third, it's just that you can't represent it properly in the decimal format.

19:27 **T:** Ok. So let me, let me go with points of agreement I have. If we try to represent a third by saying 0.3 *[T writes:* ⅓ 0,3 *]* I would say that's a pretty poor representation, right, we can get closer. But if we're trying to say it was 0.33 *[under* 0,3 *T writes:* 0,33*]* then I'm starting to say, 'Ja, that's a bit better,' right? And if you said 0.333? *[T writes:* 0,333*]* that's a bit better, right? And if you said 0.3333 *[T writes:* 0,33333333333 *]* that's pretty close to what a third actually is, right? So the mysticalness in this whole thing here *[T circles the dots after* 0,333 *]* is what do those 3 dots represent? And this is the shift your mind has to make from dealing with the finite, to dealing with the infinite.

20:27 **T:** So, just imagine that I spent enough time here that I could fill the entire board with 0, and then 3s, right? All the way to the end here. *[T points to the right hand bottom corner of the board]* I've now like two thousand 3s, right? I'd agree with you I've got very close to a third, but I haven't actually got there. So let me ask you. Would you agree with me that that's where I've got? So imagine I've managed to keep on filling this board all the way to the end here with 3s, have I got quite close to a third but not actually got there yet?

21:06 *[Class expresses agreement]*

21:07 **T**: Everybody happy with that? Ja. And even if I covered every board in every classroom in the school with 3s I would have got very close to a third but I wouldn't have got there yet. Yes? When I'm doing that am I dealing with a finite number of 3s or an infinite number of 3s?

Some learners: Infinite number of 3s.

T: So if I covered every white board in every classroom in this school with 3s, so it's 0, and then all those 3s – every… Am I dealing with a finite number of 3s or an infinite? *[T points to a learner]* Let's have some arguments

21:51 **FL6:** Finite. Because if you went around and you counted all of those you'd get a number and, ja, that number has to be finite, it can't be infinite.

T: What about… Do you agree? Disagree?

Most learners: Agree

22:03 **T:** Agree. Right. And even if I went and put 3s on every whiteboard in every classroom in every school in the whole of Jo'burg, I would still be dealing with a finite number of 3s. So every classroom in every school in every country and every… I'd still be dealing with a finite number of 3s. I'd be getting incredibly close to a third, do you agree with me? But I wouldn't be at a third yet. Right. I also haven't dealt with the infinite yet. I… So one has to get in your head, and this is the really difficult thing with infinity, there is a difference between a very, very, very, very, very, very, very, very, very big number and infinity. There is a qualitative difference. And so, when I put those three dots there *[T points to the 3 dots after the* 0,333 *that she has already circled]* I cannot just be thinking of a very, very, very, very, very, very big number. I have to be thinking of a number that goes on forever. And a number that goes on forever is different and behaves differently to a number that actually stops – even if it stops after a long time.

23:19 **T:** And now I'm going to tell you a little bit about how maths is made because this is essentially the story of infinite decimals is that infinite decimals are, occur in mathematics, occur precisely because at some point somebody was doing this *[T points to the long division sum]* exactly as you guys here were saying. They said, 'What is 1 over 3? It's 1 divide by 3. It is go and put this in and when I start doing this what do I discover?

These 3s are going to go on for ever.' And so what I want to be able to say is when I have 0,333 with the 3s going on forever – not just being a very big number of 3s – the 3s going on forever, I do want it to be a third. Because in maths we want things to be consistent. We cannot have the division giving us one answer and then we say well it's actually not really equal, right? If the division tells us that 1 divide by 3 gives us 0,333 going on forever, we want to be able to make sense of it in such a way that 0,333 going on forever is a third. Do you get what I'm saying? That mathematicians have to try and make the whole system fit together. So what they had to try and do was make this whole world consistent.

24:51 T: Now I'm going to give you another example which hopefully you'll relate to quite well of how mathematicians then make up a definition for things so that it works consistently with things they already know, because that's what we're going to do. We're going to make up a definition for infinite decimals so that it fits with all the other maths that we know. So let me show you. This is a totally unrelated thing but it's showing you another example of where mathematicians made up a definition so it fitted.

25:27 **T:** *[T writes:* 5^0 *]* What is 5 to the 0 equal to?

Some learners: 1

T: Why?

A learner: I don't know.

[laughter]

T: *[after* 5^0 *T writes* = 1 *]* Why? Why can't I say 5 to the 0 is 5? *[T writes:* $5^0 = 5$*]* Or in fact it might be quite nice to say 5 to the 0 is 0. *[T writes:* $5^0 = 0$*]* Why can't I? Ja?

ML1: Well, let's say like 5 to the 2 is 25

T: Yes

ML1: So to get that 5 to the 1 you divide by 5, right?

T: Yes

ML1: So then with that pattern, to get back to 0 you divide by 5

T: So you go here and you say you get 5 ja

ML1: So if you… So to get from the 5 to the 2, 5 to the 1 you divide by 5.

T: Yes

ML1: So logically 5 to the 1 to 5 to the 0 should also be divided by 5, which is 1.

A female learner: Ja

T: So you say 5 to the 1… Say it again. Come and do it on the board. I'm not… Make an argument for us rather, I'm not ? for us

26:20 **ML1:** *[comes to the board]* Well let's… I'm trying to put this in a pattern. So you've got 5 to the 3, ok. *[ML1 writes:* 5^3 *]* To get to the 5 to the 2 you divide by 5, right? *[next to* 5^3 *ML1 writes* $\div 5^2$ *]* So that's 5 to the 2.
[ML1 changes his sum to $\frac{5^3}{5} = \frac{5^2}{5}$ *]* Wait. So that is equal to that. How's(?) that?
[some of the class chuckle]

ML1: So with this pattern you divide by 5, you get 5 to the 1.

T: Yes

ML1: So to get 5…

[After $\frac{5^3}{5} = \frac{5^2}{5}$ *ML1 writes* $\frac{5^1}{5}$ *]*

[more laughter

26:43 **ML1:** Ok, I can't write on the board and it's not ? so far. Actually, let me just write this, ok. So it's 5. Ok, so divided by 5 you get another 5

T: Ja

[more laughter from the class]

ML1: Somebody with better handwriting should probably come.

T: ? handwriting

[class shouts out comments]

FL6: ? with exponent rules, so like you were saying 5 to the 2 divided by 5 to the 1, 2 minus 1 isn't(?) 0. Ok. 5 to the 2 divided by 5 to the 2, 2 minus 2 is 0. So 5 to the 0 equals 1 because ?

27:23 **T:** Ja. Both of you are doing very similar arguments, right? They're saying… They're giving an argument we've got some learning(?) rules, right. How do we build up exponents? We start off by saying 5 squared is just a nice short way of writing *[T writes:* $5^2 = 5 \times 5$ *]* 5 times 5, right? And then if we want to do division what do we have? Say we have 5 to the 3 over 5 to the 2, we see it's 5 times 5 times 5 over 5 times 5. Cancel, cancel. After a while we see this works out as a nice pattern of in fact what we can just do is we subtract, right?

[T writes while she talks: $\frac{5^3}{5^2} = \frac{5 \times 5 \times 5}{5 \times 5} = 5^{3-2} = 5^1$ *]*

So this is how it all ends up.

28:03 T: Then your argument comes in. Alright, so what would 5 to the 2 over 5 to the 2 be?

[I think T writes: $\frac{5^2}{5^2} = 1$ *]*

Well we know if you've got the same thing over each other it must be 1. And our exponent rule tells us that 2 minus 2 is zero… It will be 5 to 2 minus 2 which is zero.

[I think T writes: $5^{2-2} = 5^{0}$ *]*

And so this tells us if we call 5 to the zero… If we talk about something to the power of zero as anything other than 1, the whole system is going to clash with each other, do you see what I'm saying? There's nothing inherent in anything in the world that says 5 to the zero is 1. *[T points to* $5^{0} = 1$*]* But if we want the whole system of exponents to work with division as we know it that if you divide something by itself you get 1, you've got to have that 5 to the zero must be 1. And so their mathematicians said, 'Ok we must make 5 to the zero equal 1.' *[T circles:* $5^{0} = 1$ *]* You get what I'm saying?

29:05 **T:** Now there's a lot of points in mathematics where in order to make everything work together nicely, we have to define things in a certain way. *[T cleans the board]* And that is exactly what we are going to work on now.

29:29 **T:** So let me just say what we are trying to do now is we've got this funny animal that we've met *[T writes: 0,333…]* which is an infinite decimal. And this one you're comfortable with, sort of *[T writes: 0,999…]* But this one we certainly weren't very comfortable with. And what we need to do is create a definition for this, a meaning for this that allows us to work with it along with all the other mathematics that we know. So what are two of the things that we actually have that we know, right. When we get… When we do 1 divided by 3, right, we get 0.333 and these things go on forever.

[T writes (1) ⅓ 1 ÷ 3
we get 0,33333…
⅓ = 0,333… *]*

So in order to define this thing in a way that's consistent with what we know already, we would pretty much like to have that those two things must be equal to each other, right? And we also want then to extend that definition so that it works for any infinite decimal. So it must work for any infinite decimal

[T writes (2) Any infinite decimal *]*

31:05 **T:** Ok, so our job at the moment is to give a meaning to 0.333 recurring. That is what we are going to do. And I know this is very high level, so just see whether you can keep up with it as we go and then we'll give you a bit of a chance to work with it. *[T writes:* 0,333… *]* One way to think of what 0.333 recurring is is that what you've got is a whole lot of decimals that just go on, right? You've got… First of all you've got the 0.3. Then you add to it… *[after* 0,333… *T writes* = 0,3 + 0,03 *]* the second place. Yes? Then you add to it the third place. *[T adds on* + 0,003 *]* Ja? And then we keep on doing that for ever and ever. *[T adds on* + *]* Does that feel ok? So that seems like one step in our definition that we can just say it's adding all these little things together.

[Thus it looks like: 0,333… = 0,3 + 0,03 + 0,003 + *]*

32:13 **T:** Ok. Let's get out of decimals and into fractions. What's that as a fraction? *[T points to* 0,3*]*

A learner: Three tenths

T: Three tenths, nice and easy. *[T writes:* $= \frac{3}{10} +$ *]* This one? *[T points to* 0.03*]*

3 over 100 *[T writes:* $\frac{3}{100} +$ *]* This one? *[T points to* 0,003*]* 3 over?

A learner: A thousand

T: A thousand. *[T writes:* $\underline{\quad 3}$ + *]* And we repeat, it's ongoing, right?

1000

[Thus it looks like: $= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$ *]*

32:36 **T:** Now this then keeps on going for ever and ever is a problem *[T points to the space after the plus sign]* Right? It's the forever and ever I can't deal with. I can deal with anything other than the forever and ever. So let's stop things short and just deal with things up to a certain point. So I'm going to use this terminology here. *[T writes:* S_n *]* and I'm going to say Sum up to n of them, right. So let's just get the terminology straight. *[T writes:* $S_1 =$ *]* If I say just sum using one term I need just take three tenths, right? *[after* $S_1 =$ *T writes:* $\frac{3}{10}$ *]*

Say I'm using two terms will be *[T writes:* $S_2 = \frac{3}{10} + \frac{3}{100}$ *]*

Yes, you get what I'm using as a terminology? If I said take 3 terms I'd go three tenths, plus three hundredths, plus three thousandths, right? Ja. Everybody happy?

[T writes: $S_3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$ *]*

[T writes: $S_n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$ *]*

33:48 **T:** Now all our decimals, I mean our exponents, *[points to the above equation]* as you can see this is 10, 10 squared, 10 cubed. Right? So what will the nth term be? Can anybody tell me here. *[to the last equation T now adds:* $+ \ldots + \frac{3}{10}$ *]*

What will I have to put? It's 10 to the power of?

A learner: n

T: n. Everybody happy? *[the last term now becomes* $+ \frac{3}{10^n}$ *]*

You first term is 10 to the 1, your second term 10 squared, third term 10 cubed. So your n term is going to be 10 to the? n.

[under this equation I think T writes: $S_3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$ *]*

$$\quad 10^1 \quad 10^2 \quad 10^3$$

34:27 **T:** So we've got a sum to n terms. That I can deal with. *[T cleans portion of the board]* I now want to add all these up because what I'm going to get to, what I'm trying to do is to say what I want to have is to get to the point where you go on for ever and ever. *[T points to* 0,333…*]* Right? So what am I going to try to do? I'm going to get an answer for the sum and then I'm going to say what happens, and I'm going to define what happens when I let n go on for ever and ever, get bigger and bigger. Is this making sense or is this too… I'm seeing many lost faces here. Too lost? Nothing. You're ok. One person's ok. Anyone else? You're with me still? *[T puts a thumbs up]*

35:15 **T:** Ok, we'll carry on. You notice I was not going to try this, but you guys pushed so much further last time than we intended to go, you got all the arguments together that most other people don't get to get to, we thought we'd see if you could follow the most abstract argument. So let's work on this.

35:37 **T:** Alright. Here we go. We need to add up this thing. Have you done any… *[addressing the class teacher?]* Have they done any of this the geometric? Ok, so we'll do it, we'll do it in full. There's a cute little way that allows us to get, to add those things together and I'll show you what it is. What we do is we're going to multiply our sum by 10. *[T writes:* 10 S_n = *]* So we're gonna multiply this whole thing by 10. If we multiply 3 over 10 by 10 what do we get? *[T points to the* S_n *sum]*

A learner: 3

T: Right. *[T writes:* 3*]* And if we multiply 3 over 100 by 10 what do we get?

A learner: 3 over 10

T: 3 over 10. *[T writes:* $\frac{3}{10} +$ *]*

And if we multiply 3 over 1000 by 10 what do we get?

A learner: 3 over 100

T: 3 over 100, right. *[T writes:* $\frac{3}{100} + \ldots +$ *]* And we keep on going.

If we multiply 3 over 10 to the n by 10 what do we get? *[T writes:* $\underline{3}$ *]*

[Learners try and work this out]

36:50 **T:** Ok, everybody you can do this. 3 over 10 to the n multiplied by 10 to the power of?

[T writes: $\frac{3}{10^n} \times 10^1$ *]*

A learner: 1

T: 1. So what does it give you?

[learners make comments]

T: If you need to get out a piece of paper, do.

[learners continue to make suggestions]

37:20 **T:** *[points to ML8]* Ok?

ML8: ? cancel down.

T: No, careful, right. You're not going to cancel because this is, this is not… It's 10 lots of n, right? n lots of 10, right? What does this mean? It means you've got 10 times 10 times 10 times 10, right, n times. *[T writes:* 10 x 10 x 10 x *]* So you're just going to get rid of one of those n 10s. So how many are you left with? *[T points to a learner]* I think there was someone over here who gave it to me.

Female learner: 3 over 10 to the n minus 1

T: Ok. She said 3 over 10 to the n minus 1, right. *[T writes:* $\frac{3}{10^{n-1}}$ *]*

because you have n10 sitting there and you got rid of one of them so it's 3 times 10 to the n minus 1. *[T addresses ML1]* Yes?

ML1: But if n continues into infinity surely you can't

T: No, and we're not. That's why I said we're leaving infinity alone, right. Because we're leaving infinity alone for now because we don't know how to deal with it. That's why we're going to a finite number n. We're going to sit with our finite number n and then we're suddenly going to go to infinity. So let's sit with our finite, right.
[T continues to add to the 10 S_n sum*:* $\frac{3}{100} + \ldots + \frac{3}{10^{n-1}}$ *]*

[The sum is: $10\,S_n = 3 + \frac{3}{10} + \frac{3}{100} + \ldots + \frac{3}{10^{n-1}}$ *]*

So this is like n is a million or something, it's absolutely finite, right.

38:39 **T:** Now what we can do, can you see here that everything that's here *[T points to the* S_n *sum]* is here *[T points to the* $10S_n$ *sum]* except for… *[T circles the* 3 *in the* 10 S_n *sum]* you've got that 3 here and you've got that on the top
[T circles the $\frac{3}{10^n}$ *in the* S_n *sum]*

Because the term just before this on the top would be what? What would have been sitting here? 3 over?

[T points to: $S_n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots \frac{3}{10^n}$ *]*

A learner: 10 n minus 1

T: *[After the dots T adds in* $+ \frac{3}{10^{n-1}}$ *]*

[T nods] Because it would have been the one just before the 10 to the n term, right? So everything is repeated except you've got this *[T points to* $\frac{3}{10^n}$ *]*

and you've got this *[T points to* 3 *]*, right? So let's do a subtraction. Let's say 10 to the… 10 Sn minus an Sn we will have 3 left here and then we will have to take off 3 over 10 to the n there.

[T writes: $10\ S_n - S_n = 3 - \frac{3}{10^n}$ *]*

39:45 **T:** Yes. Alright. A little bit of algebra. Take out the Sn we've got 10 minus 1 and here I can take out a 3, right, and I've got 1 minus 1 over 10n, right.

[T writes: $S_n\ (10 - 1) = 3\ (1 - \frac{1}{10^n})$

Female learner: Then you could, ?

T: Yes? What's someone at the back said? Have you noticed something?

Female learner: I think so.

T: Yes, tell us.

Female learner: ?

T: What's 10 minus 1?

Learners: 9

T: You can take that and divide it, right. So we're left with our sum is equal to 3 over 9 you have cancelled(?) it straight away to a third. One over three. Right.

[T writes: $S_n = \frac{1}{3}\ (1 - \frac{1}{10^n})$ *]*

40:36 T: Alright. Now we've got to keep the ?, we've got to keep pausing with this. This is probably a long argument and we've got to just see what we doing. We're trying to get a definition in the end for what 0.33 recurring is. *[T points to* 0,333… *]* Right. And we're trying to work that definition so that it makes sense with everything we know already in maths. So we're going back to what we know already which is to leave the infinite alone and go back to the finite and work with the finite and then we're going to build up to the infinite.

41:08 **T:** So what we're seeing is if we take any of these 0.33s when you've got a finite number of 3s *[T writes:* $= 0,\underbrace{333}_{\text{number of 3's}}^{n}$

[Thus: $S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = 0,\underbrace{333}_{\text{number of 3's}}$ *]*

Right. So it's a finite number of 3s. If I put n 3s here it gives me the sum one third one minus one over 10 to the n, right. *[T points to this on the board]* As long as there are n 3s on the right. Ok. Do you see… What does this tell us? Why is 1 over 10 to the n?

[T points to: $\frac{1}{10^n}$ *]*

If you picture it, if n's 1 it's just 1 over 10, right? If n's 2 it's 1 over 100, if n's 3 it's 1 over 1000. Can you see it's just a little fraction – one tenth, one hundredth, one thousandth, one millionth – however big it is, right? But any time you get this what are you saying?

[T points to: $S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = 0,333$*]*

You're saying you've got one third and you're subtracting off that tiny little fraction. So any of these 0.3333 things, as long as you've got a finite number of 3s is just a teeny bit little (let me try and get my teeth around this) a teeny little bit less than a third. Do you agree with me?

Some learners: Yes

42:39 **T:** Yes. So what we have is we have *[T cleans part of the board]* a whole sequence of little things like this. 0.3 which is smaller than 0.33 which is smaller than 0.333 which is smaller than 0.3333 etc etc. All of them being smaller than a third.

[T writes: $0{,}3 < 0{,}33 < 0{,}333 < 0{,}3333 \quad < ⅓$ *]*

Just a teeny weeny little bit less than a third, provided we are living in the finite world.

43:19 **T:** And then we can deal with all of that. We can deal with that as long as we are living in that finite world. What we want to do is now make sense and make a definition for this *[T points to* 0,333… *]* that works. A definition for the infinite world. And the definition for the infinite world is gonna be… If we want to have an infinite world we're gonna have to take Sn when n is infinitely large. *[T writes:* S_n when n is infinitely large*]* So this is our move into the infinite world, right. In the finite world what we have is that all of these things are just a teeniest little bit less than a third. Now you want to make a definition in the infinite world that will work and that will fit with everything that we know. So what we want is we want to make a definition. We are going to say what it means when we take n being infinitely large. And what is the only option we can take? What is the only number we can give that will work here? The only thing that we can say that will work is we say that when n gets infinitely large this thing has to be equal to one third. And why is it?

44:57 **T:** In other words when we, when I'm talking about Sn when n is infinitely large I am saying I'm talking about that.

[T writes while she speaks: When n is infinitely large
S_n

$0{,}333 = \frac{1}{3}$ *]*

Why do I say the only possibility we can have is that it must be equal to a third? Because if I look here

[T points to: $S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = 0{,}333$*]*

it can't get bigger than a third, right? Because for each one of these little things I have just shown it's one third minus a little less. So for every single one of the finite ones it's a third minus a little piece. *[addresses ML1]* Yes?

45:40 **ML1:** But so like you're saying it's infinitely close so it must be the same?

T: That's exactly what I'm saying.

ML1: But surely then you could argue that because you can find infinity between any two numbers it's also infinitely differences…

T: Say it again, because I can find…?

ML1: Well like between… Infinity itself is an unimaginably huge number.

T: Yes

ML1: But you have infinity between 1 and 2, 2 and 3 because you can get an infinite amount of numbers between them

T: You can

ML1: So surely the fact that it's infinitely close you can also argue that it's also an infinitely small difference, but the difference is still there.

46:19 **T:** No, let's just… I mean I can see what you're trying to say, but if I try to define it as being something that was close to a third but not quite there, right? So if I… So then it must… I must subtract off some little thing, ok, whatever it is, but some little thing, right?

[T demonstrates on the board: $\frac{1}{3}$ – 00000001 *]*

If I take off… If I take one third minus some little thing it will be equal to one of those finite sums. And that's where I'm going to have a problem because when I get to the infinite sum it has to be bigger than all those finite sums, right?

[T points to: 0,3 < 0,33 < 0,333 < 0,3333 < ⅓ *]*

because it has more 3s than any finite sum has. So it has to be bigger than all of them. So if I say it's one third minus a little piece it's going to be equal to one of the little sums – the finite sums – and therefore it's going to be not the biggest(?) of all of them(?) and not…

47:34 **ML1:** But even if it's an infinitely small piece?

T: Ok, then you question is what do you mean by an infinitely small piece?

ML1: Well I mean in the same way that you can… Well, like this. So like back to the other thing.

T: Ja

ML1: So well 0,9 plus 0,1 is 1

T: Yes

ML1: That 0,9 that relationship continues on.

T: Ja

ML1: Surely it continues on for infinity? So you have that 0,1 regardless of how many zeros, it will be an infinite amount of zeros.

T: You're asking… I mean you see what… What you're doing is you're taking us to the nub(?) of what we are doing here, right. *[T writes: 0,333…]* What mathematicians had to do was they said, 'We got to a point where having numbers that go on for ever and ever has occurred. It's occurred so we have to deal with it. We have to try and see and give a meaning to it that makes sense, right?' And they could have chosen many many ways to give meaning to it. The way they chose to give meaning to it is through this, through this infinite sum idea *[T points to the board]* and through seeing it as being the biggest of all those finite sums. And for it to be bigger than any of the finite sums it had to be a third.

48:52 **T:** You are saying, 'Ok what about this idea of an infinitely small thing? Because if I could have an infinitely small thing then actually your argument would make sense. Let me just finish this and I'll come back to you. So you're right. What you're saying is if we could talk about your thing with an infinite expansion in the middle *[T writes:* 0,000 … 0,0001 *]* you could then have a totally different argument to what I'm giving and you're absolutely right. But then at this point what you are doing is creating another definition of maths which is going to create another system of maths. And the very first thing we have to do is define what this means, right? Because at the moment it's a hand wave. It's… There's a whole lot of things in here that go on forever and your friend over here is going to say that it ends, so it's not infinite. You are going to have to pin down a definition of this that doesn't involve waving your hand. And you can pin down the definition of this. And you can create a totally different mathematics and a totally different system in which 0.333 recurring will not be equal to a third. And in which all the rules of maths as we currently know them will not work. And there are people and there are mathematicians who have done that and who have had exactly your idea and who did exactly what you did. And that is they came up with a very different system. So this is the point. That in fact the reason why it's a third is because we made it to be a third so that the rest of mathematics as we know it would work with it. *[addressing ML1 again]* Ja?

50:40 **ML1:** Ma'am my thing was just because with that reasoning the reason I asked was I the 0,3 the one just below will argue that. The one that the biggest one of thos must be a third.

T: Mmm

ML1: But surely saying 'it's the biggest' makes it finite? Because can't you get… Can't you add like one more part? I don't know… Infinity is just infinite.

T: Yes

ML1: Infinity… Saying it's the biggest pins it down which makes it finite. Does that make sense?

[Learners make comments]

51:11 **T:** *[laughs]* No, it's… It is the biggest and it is a third, but yes, you cannot pin it down and write down exactly how many 3s after the decimal comma there are. *[T addresses FL4]* Ja?

FL4: I think you can say it's a third because… I don't know a better word, but it's like the third more of an infinite. So it's ? But I mean it's not dividing yet. So…

ML1: True

FL4: It's, it's like it has just tried to write it down again, it's just like making it easier to understand.

ML1: I don't(?) understand.

51:55 **T:** And then if we push this, right, because what we wanted was an argu… was a definition for an infinite decimal which would work for anything. So we can change this entire thing that we've done with a third to work exactly… or 3s… to work exactly the same with 9s. So we would just replace all the 3s here by 9s.

[T replaces the 3s with 9s: 0,999… = 0,9 + 0,09 + 0,009 +

$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots \quad]$$

Right? Happy that if this is the way we choose to define decimals we say they are adjusting(?) these sums so the sum can be written like that. We can do exactly the same process as we did here to get the answer for the sum

[T continues replacing 3s with 9s: $S_n = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots \frac{9}{10^{n-1}} + \frac{9}{10^n}$

$10 S_n = 9 + \frac{9}{10} + \frac{9}{100} + \ldots + \frac{9}{10^{n-1}}$ *]*

52:54 **T:** And so let's think what would we get here. We would get 10 Sn minus Sn again but here we would have a 9 and here we would also have 9.

[T changes the 3s to 9s: $10 S_n - S_n = 9 - \frac{9}{10^n}$ *]*

And so that would come out as 9

[T changes 3s to 9s: $S_n (10 - 1) = 9 \left(1 - \frac{1}{10^n}\right)$

And this would come out as a 1

[$S_n = 1 \left(1 - \frac{1}{10^n}\right) = \underset{\text{finite number of n's}}{0,999}$*]*

So we would say that any of these ones, any of these sums where we've got a finite number of n's is going to be just a teensiest weensiest little bit smaller than 1. And what we get is we get exactly this *[T cleans part of the board]* - a whole series of them. This where n is 1, this where n is 2, this where n is 3 etc etc… until we get each and every one of them has to be smaller than?

[T writes as she speaks: 0,9 < 0,99 < 0,999 < 0,9999 < 1 *]*

A learner: 1

54:09 **T:** 1. What do we define it to be when n is infinitely large? We say, same as we did there, when n is infinitely large it must be that number *[T points to* 1*]* that these things are approaching *[T points to* 0,9 < 0,99 < 0,999 < 0,9999 *]* So we define 0.999 recurring to be equal to 1. And that's why it is. Alright? And so the thing that is difficult in one's head is… And the thing that I think that has been… we see most people thinking is when they see 0.999 recurring *[T writes:* 0,999…*]* what you're thinking is that you have a whole lot of 9s. Many, many, many, many 9s, but you're still thinking a finite number of 9s. What mathematicians had to do was say, 'What and how must we define this thing when we move over into infinity?' And the only way they could come up with defining it was saying you've got this whole series of them that are there for the finite numbers.

[T points as she speaks: 0,9 < 0,99 < 0,999 < 0,9999 < 1 *]*

55:30 **T:** And what you've got to try and do is say well what can the infinite then be? The only thing the infinite thing could be that would make sense with all of this was 1. Because if you try to define it to be something just a little bit less than 1 it would coincide with a finite n. Do you get what I'm saying? So you said it was just a little bit less than 1 it would be something like 0.999999999 with a million 9s, but that's still finite, right? So it couldn't be that. And even if you tried to put in a million 9s it couldn't be that because that's still finite. So the only thing it could be to make sense was to say it is equal to 1. And it's quite ok then for it to be equal to 1 and it's because these three mystical(?) dots don't just mean an awful lot of 9s, they mean infinitely many 9s. It doesn't mean very big, it means infinitely big.

56:29 **T:** I think our time is up, hey? I wanted to ask questions.

D: To quarter past

T: Oh, so we've got to quarter past. Ok, did you have your sheets you wanted…

<u>End of this video</u>

APPENDIX C

PARTICIPATION LETTERS

Participation in Research Study

Dear learner,

My name is David Merand and I am currently studying for a master's degree at the University of Witwatersrand and am doing research on mathematical education.

My research topic is: **A study of Grade 11 learners' understanding of concepts related to infinity.**

I will be looking at various questions related to infinitely recurring decimal fractions and different ways of representing them during a classroom session. If you agree to participate, you will be working in groups to solve various problems and to discuss these with your teacher and class.

I am inviting you to take part in this study and would like permission to record information while you are working in the classroom and during the interviews with your group of learners. These recordings will be done by means of video taping of the classroom sessions and audio (sound) taping of the interviews.

I will be arranging two 80 minute classroom sessions with your grade 11 teacher and his/her advanced program mathematics class to conduct lessons which will be given by your teacher on a pre-arranged topic related to infinitely recurring decimal fractions. These sessions will be followed by a 10 minute audio recorded interview with each working group of learners. These sessions and interviews will be scheduled during school hours.

I undertake to maintain anonymity and confidentiality of yourself and the school in all my research writing about the study. After three to five years all data related to the study will be destroyed.

There are no foreseeable risks in participating in the study. You will not get paid for participating. Any information relating to the study will have no impact on your school marks or other assessments.

I trust that you will accept this invitation to participate in the study. You are of course, free to withdraw permission for data to be collected about yourself or to used for research at any stage along the way without any impact on you or your school results or marks. If you are not willing to participate your comments and written work will not be used or referred to in the study.

I very much hope that this research project will ultimately have benefits for you and your teacher. Please do not hesitate to contact me (0836100995) if you require further detail or clarification.

Best wishes,

David Merand

Division of Mathematics Education
Wits School of Education
david.merand@wits.ac.za

Participation in Research Study

Dear Parent/Guardian,

My name is David Merand and I am currently studying for a master's degree at the University of Witwatersrand and doing research on mathematical education.

My research topic is: **A study of Grade 11 learners' understanding of concepts related to infinity.**

I will be looking at various questions related to infinitely recurring decimal fractions and different ways of representing them during a classroom session. The participants will be working in groups to solve various problems and to discuss these with their teacher and class.

I am writing here to formally ask for your written consent to collect the following data of your child during classroom sessions:

-video taping of classroom observations
-audiotaping of learner discussions and during interviews

I would like to arrange two 80 minute classroom sessions with your child's grade 11 teacher and his/her advanced program mathematics class to conduct lessons which will be given by the teacher on a pre-arranged topic related to infinitely recurring decimal fractions. These sessions will be followed by a 10 minute audio recorded interview with each working group of learners. These sessions will take place during school hours.

I undertake to maintain anonymity and confidentiality of your child and the school in all academic writing about the study. After three to five years all data related to the study will be destroyed.

There are no foreseeable risks in participating in the study. Your child will not get paid for participating. Any information relating to the study will have no impact on your child's school marks or other assessments.

I trust that you will accept this invitation for your child to participate in the study. You are of course, free to withdraw permission for data to be collected or used for research at any stage along the way without any impact on you or your child's school results or marks. If you are not willing for your child to participate any comments or written work done by your child will not be used when transcribing the results of the study.

I very much hope that this research project will ultimately have benefits for your child and his/her teacher. Please do not hesitate to contact me (0836100995) if you require further detail or clarification.

Best wishes,
David Merand

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